International Journal for Numerical Methods in Engineering

A comparative study for discharge-drawdown relationship of eccentric wells in unconfined aquifer

Journal:	*International Journal for Numerical Methods in Engineering*
Manuscript ID:	NME-Mar-15-0216
Wiley - Manuscript type:	Research Article
Date Submitted by the Author:	27-Mar-2015
Complete List of Authors:	Elmazoghi, Hasan G; University of Benghazi, civil engineering karakale, vail; marmara university, civil engineering Bentaher, Lubna S; University of Benghazi,
Keywords:	Finite difference methods, Fluids, Differential equations

A comparative study for discharge-drawdown relationship of eccentric wells in unconfined aquifer

Hasan G. Elmazoghi[1], Vail Karakale and Lubna S. Bentaher[2]

[1]Assistant Prof., Civil Eng. Dept., Univ. of Benghazi, Libya, hasan.elmazoghi@benghazi.edu.ly
[2]Prof. Dr., Civil Eng. Dept., Marmara University, Istanbul, Turkey, vail.karakale@marmara.edu.tr
[3]Research assistant, Civil Eng. Dept., Univ. of Benghazi, Libya, lina7981@yahoo.com

ABSTRACT

A comparative study between the results of the available experimental and analytical solutions with those obtained from the developed numerical model of eccentric wells operating in circular island unconfined aquifer under steady state flow conditions was carried out. The results of all the previous solution methods were expressed graphically by drawing the discharge-drawdown relationship for such wells. The objective was to define the drawdown range where these solutions approximate each other and to test the validity of the analytical solution. In contrast to the experimental model which showed that the analytical solution is only valid for small drawdowns, the present study indicated that the numerical and analytical approaches are virtually identical for all values of drawdowns provided that Darcy's law is applicable. On the basis of this comparative analysis, any of the analytical or the numerical solutions could be used in order to adequately describe the drawdown for a well placed eccentrically in a circular island unconfined aquifer under steady state conditions, although the numerical approach may have some implementation advantages over the analytical approach.

Keywords: drawdown, eccentric well, unconfined circular aquifer, numerical model

1 INTRODUCTION

Several researchers have studied well operating systems in circular island aquifers under steady state flow conditions using theoretical and experimental methods. Dupuit (1863) studied the hydraulics of flow to a well and derived an expression by neglecting the effect of the seepage face (the vertical surface of the aquifer which is exposed between the water level in the well and the location at which the phreatic surface intersects the well opening) and ignoring the flow in the unsaturated zone above the phreatic surface. In 1948 Babbitt and Caldwell constructed a circular island laboratory model in order to test the validity of Dupuit (1863) and Muskat (1937) equations for well interference in circular unconfined aquifers under steady state flow conditions. They concluded that these theoretical equations are valid for a small group of wells. Hantush and Jacob (1960) presented a theoretical study under steady state flow conditions for a discharging well located eccentrically in a circular confined aquifer along whose outer cylindrical boundary either the drawdown or the flux is zero. De Wiest (1963) obtained an analytical solution for the flow to an eccentric well in a leaky circular aquifer with lateral replenishment, both for steady and unsteady cases. De Wiest gave graphs to show the influence of vertical leakage and lateral replenishment on the relationship between drawdown at the well and eccentricity. Later on, Verruijt (1982) presented a theoretical equation for the wells eccentrically located in unconfined circular aquifer under steady state flow conditions. Recently, Birpinar and Gazioglu (2002) performed an experimental study for wells located eccentrically in an unconfined aquifer in order to test the validity of the equation presented by Verruijt. Their study showed that Verruijt analytical equation is only valid for small drawdowns. Phoolendra & Kristopher (2013) stated that despite the many advanced solution methods available, there still exists a need for realism to accurately simulate real-world aquifer tests.

In the present paper a steady state two-dimensional numerical model, using the finite difference method, was prepared with the wells eccentrically located in a circular island unconfined aquifer. A comparison of the results of the present numerical model with each of the analytical solution by Verruijt (1982) and the experimental results of Birpinar and Gazioglu (2002) was considered. The objective was to define the drawdown ranges where these solutions approximate each other and to test the validity of Verruijt (1982) equation. The bases for comparison of these models are the predicted drawdown and the discharge-drawdown relationship at the well in operation under different discharge rates. Understanding of such relationships is of practical interest in operating unconfined radial flow systems. The conditions assumed in this study are: the aquifer is homogeneous, the wells fully penetrate the aquifer and are unlined, Darcy's law is valid for the flow in the aquifer, only one well is pumped at a time and the well is pumped at a constant rate until equilibrium conditions.

2 GOVERNING EQUATION & BOUNDARY CONDITIONS

Equation (1) represents the two dimensional partial differential equation for incompressible steady groundwater flow in a homogeneous unconfined aquifer, which can be obtained after simplification and arrangement as Poisson's equation. This equation was formulated by introducing Dupuit's assumption that heads, h, do not vary in the vertical direction (i.e., $\partial h / \partial z = 0$) resulting in horizontal flow (Wang and Anderson, 1982).

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{2R(x,y)}{k} \tag{1}$$

Where $v = h^2$, $R(x,y)$ is the volume of water recharged per unit time per unit aquifer area and $R(x,y) = Q/\Delta x \Delta y$, where Q is the recharge rate from the well and k is the permeability coefficient. In order to obtain the numerical solution of equation (1), which governs the water flow in the present study, one must express it altogether with the boundary conditions in terms of its finite difference approximation. For this purpose, the total flow domain is discretized by superimposing a mesh-centered grid network with irregular spacing between the nodes located at the intersections of the grid lines, see Fig. 1. The finite difference form of Poisson's equation at any node (*i,j*) is given by equation (2).

$$v_{i,j} = \frac{Av_{i+1,j} + Bv_{i-1,j} + Cv_{i,j+1} + Dv_{i,j-1}}{E} + \frac{R(x,y)}{Ek} \tag{2}$$

Where $A = \frac{1}{\Delta x_{i+1,j}(\Delta x_{i+1,j} + \Delta x_{i-1,j})}$, $B = \frac{1}{\Delta x_{i-1,j}(\Delta x_{i+1,j} + \Delta x_{i-1,j})}$,

$C = \frac{1}{\Delta y_{i,j+1}(\Delta y_{i,j+1} + \Delta y_{i,j-1})}$, $D = \frac{1}{\Delta y_{i,j-1}(\Delta y_{i,j+1} + \Delta y_{i,j-1})}$, and $E = A + B + C + D$

There will be one equation of the form of equation (2) for each node in the flow domain. The finite difference method approximates Poisson's equation by N linear algebraic equations involving N unknown values of head. The boundary condition of the problem considered in the present study is a Dirichlet condition. That is, the head is known (h = constant = 42.8 cm) for all the nodal points on the outer circular boundary of the flow region and remains unaffected for distances greater than the radius of the aquifer. The solution of these equations is first obtained in terms of v and the heads (h) at all nodal points are then obtained by taking the square root of v. Previous researches by Boulton (1951), Hantush (1964), and others proved by mathematical procedures that the discharge obtained by introducing the Dupuit assumption represents a correct solution eventhough the water table obtained by the same procedure, known as Dupuit parabola, significantly deviates from the real water table, especially where the streamlines are strongly curved. They suggested that the deviation from the

Dupuit discharge is within 1 to 2%. On the other hand, Shamsi and Narasimhan (1991) suggested that the Dupuit model underestimates the discharge because the flux through the seepage face is ignored and the deviation may range up to 10 to 12%. Siddiraju (2013) concluded that the specific capacity of open wells increases as the diameter of the well increases.

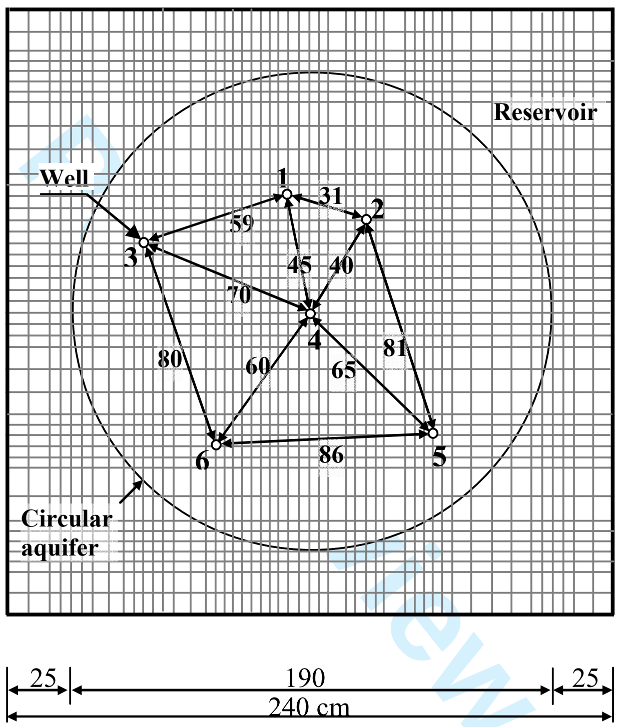

Fig. 1 Finite difference grid superimposed over Birpinar and Gazioglu circular island physical model

3 MODEL OF THE STUDY

Birpinar and Gazioglu (2002) have given an experimental study for the circular island unconfined aquifer model under steady state flow conditions. They constructed a sand model in the laboratory where the island is contained in a square box (Fig. 1). The thickness of the circular aquifer model was 50 cm. The boundary condition head in the sand model was kept constant at about 8 cm below the sand surface. In brief, the constant values for the applied circular aquifer model were taken as: H = 42.8 cm (referred to the base of the aquifer), k = 0.468 cm/sec, r_w =1.75 cm and r = 96 cm, where H is the height of initial water table, k is the permeability of the unconfined aquifer, r_w is the well radius, and r is the radius of the aquifer (radial distance from the center of the aquifer to its external circular boundary). In order to have a direct, logical and easy comparison between the results of the available experimental and analytical solutions with the present numerical solution, the considered flow is discretized with mesh-centered nodes whose spacing are irregular and selected in such a way that the wells are exactly placed at the same positions within the aquifer as that proposed and used by Birpinar and Gazioglu (2002). In other words, the performances of the three models are compared for the same soil properties, problem dimensions and flow geometry. Using the image well theory, Verruijt (1982) developed an analytical equation for a sink (+Q) placed eccentrically at $x = p$, $y = 0$ as shown in Fig. 2, where p is the well eccentricity. The problem is solved by inserting a source (-Q) at $x = r^2 / p$, $y = 0$, where both sink and source are operating in an infinite circular unconfined aquifer. With application of appropriate boundary condition, Verruijt obtained that

$$h_w^2 - H^2 = \frac{Q}{\pi k} \ln\left(\frac{r_w r}{r^2 - p^2}\right) \tag{3}$$

When the eccentricity p = 0, equation (3) will reduce to Dupuit's expression derived by integrating Darcy's law for steady flow around a well in a homogeneous and isotropic unconfined aquifer. In the mesh centered grid used in this study, the axis of the well at (*i,j*) is assumed to be centered within the infinitesimal area $\Delta x \Delta y$, see Fig. 3. Withdrawal from the aquifer is prescribed at the well in operation, however, $R(x, y)$ is set equal to zero outside the infinitesimal area containing the well. Thus $R(x, y)$ is completely defined throughout the problem domain.

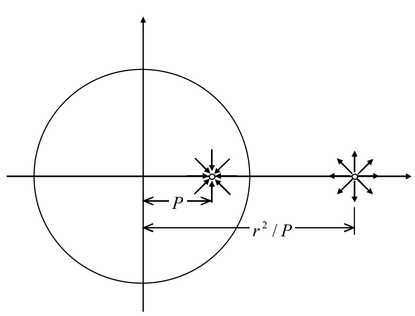

Fig. 2 Schematic sketch for Verruijt sink and source in circular aquifer

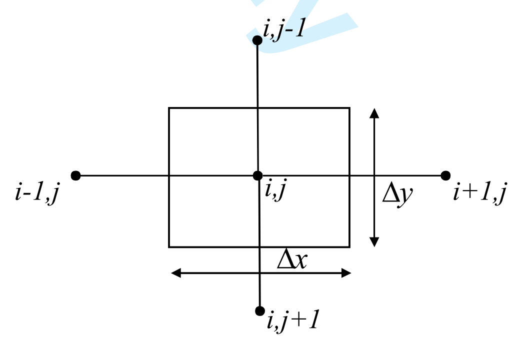

Fig. 3 Schematic sketch for the infinitesimal area $\Delta x \Delta y$ with the well at *i,j*

The finite difference computer program used for solving equation (2) is very similar to that used by Wang and Anderson (1982) for well drawdown in unconfined aquifer; however, it is slightly modified in order to match the properties and geometries of the present study. one of the main concepts in numerical modeling is verification of the model results. Some general areas of uncertainty in numerical models include: choise of mathematical model, spatial and temporal descretization, and uncertainty of boundary and initial conditions and approximation of such conditions. The uncertainty (error) of the model is some function of all of these and can be evaluated by comparing the numerical results against known analytical solutions. In this paper the developed numerical model is seen to approximately reproduce the results of similar solved examples given by Wang and Anderson (1982) and Rushton (2003). The finite difference grid is refined close to each well in operation and near the aquifer boundaries in order to minimize the mesh discretization error. The solution of the resulted linear algebraic equations is obtained by Gauss-Seidel iteration with successive over relaxation. The error tolerance (which is the maximum change in head at any node from one iteration to another) was set equal to 0.01 cm. If the water surface in any well is drawn down below the base of the aquifer, a negative value of v would have existed at the location of the well. In such cases, the head at the well is set equal to the base of the aquifer.

4 RESULTS

Six series of numerical experiments are performed in order to analyze the details of the discharge-drawdown relationship of the simulated wells in such a circular unconfined aquifer. One series is for the centric well and five series are for the eccentric wells. In each series, different discharge values are inserted in the model at the well in operation and the corresponding drawdowns are computed at that well. An example for the drawdown calculation using Verruijt analytical equation altogether with the experimental and numerical results is presented in "Table 1" for well number 1. Knowing that p is the well eccentricity and $s = 42.8 - h_w$, where s is the drawdown, then $h_w^2 - 42.8^2 = \frac{Q}{0.468\pi}\ln(\frac{1.75x96}{96^2 - 45^2})$. The calculated analytical drawdowns are shown in Table 1 for various discharge rates and compared against each of the numerical and experimental results.

From “Table 1” it is seen that the percentage differences between the analytical and experimental results are considerable and increase as the discharge rate increases. They range from 27.8% to 48.6%. On the contrary, a very close agreement is reached between the results of the analytical solution and the present numerical model.

Table 1 Drawdown values of well number 1

Q (cm^3/se)	Experimental drawdown, s1 (cm)	Analytical drawdown, s2 (cm)	Numerical drawdown, s3 (cm)	error % between s1 & s2	error % between s2 & s3
270.0	6.5	9	9	27.8	0
452.0	11.5	16.8	16.8	31.5	0
643.0	17.5	29.1	29	39.5	0.3
716.58	22	42.8	42.8	48.6	0

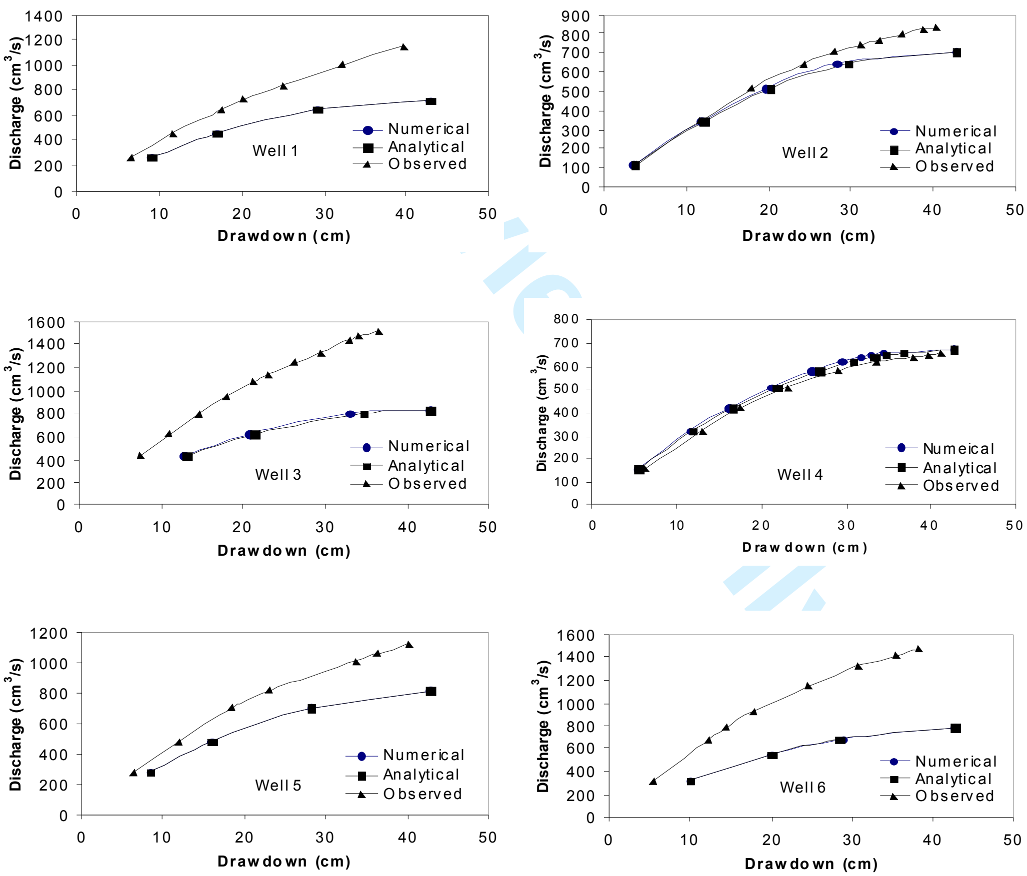

Fig. 4 Discharge-drawdown relationships for numerical, analytical and observed models

Similar conclusions (as for well number 1) are obtained for all the other five wells in this model, see Fig. 4. From this figure, and compared to the experimental model for the same discharge rate, it is observed that the analytical and numerical models considerably overestimate the drawdowns for all wells except well number 4. "Table 2" presents the range of percentage differences between the analytical and each of the experimental and numerical drawdowns for all wells. Compared with the analytical results, the maximum percentage difference in drawdown values as obtained from the present numerical model is small (approximately 3.7%); however, it is high (approximately 66%) when compared to those obtained from the experimental model. Based on such big difference, Birpinar and Gazioglu (2002) stated that equation 3 is valid only for small drawdowns. Of importance is that when equations (2) and (3) are applied to an unconfined aquifer, it is assumed that the flow is horizontal, the equipotential lines are vertical, the horizontal hydraulic gradient equals the slope of the water table and Darcy's law and the conservation of mass generally govern the groundwater flow. Darcy's law does have limits on its range of applicability and these limits must be evaluated in any application (Todd 1980). Therefore, such differences between the analytical (or numerical) and experimental solutions may be attributed to the application of a model based upon Darcy's law to media or environment where Darcy's law is inappropriate, or the use of a two-dimensional model where significant flow occurs in the third dimension. The applicability, or usefulness, of a model depends on how closely the mathematical equations approximate the physical system being modeled. For this reason, it is necessary to have a thorough understanding of the physical system and the assumptions embedded in the derivation of the mathematical equations. In order to calculate the drawdowns by the analytical and numerical methods, the discharge values presented in "Table 2" are considered as the maximum permissible discharges at which the piezometric head is equal to zero or the drawdown is equal to 42.8 cm (these values are valid for the proposed model dimensions).

Table 2 Drawdown error % and maximum permissible discharges from different models for each well

Well No.	Range of error % between s1 & s2	Range of error % between s2 & s3	Maximum permissible discharges (cm^3/s)
1	27.8 - 48.6	0 - 0.3	716.58
2	2.9 - 34.6	0 - 3.3	705.80
3	42.2 - 61.0	0 - 2.8	829.27
4	5.7 - 12.4	0 - 3.7	672.18
5	22.6 - 46.2	0 - 0	793.37
6	45.5 - 66.1	0 - 1.3	767.06

Three sets of discharge-drawdown relationships from the experimental, analytical and numerical solutions are drawn for all wells as shown in Fig. 5.

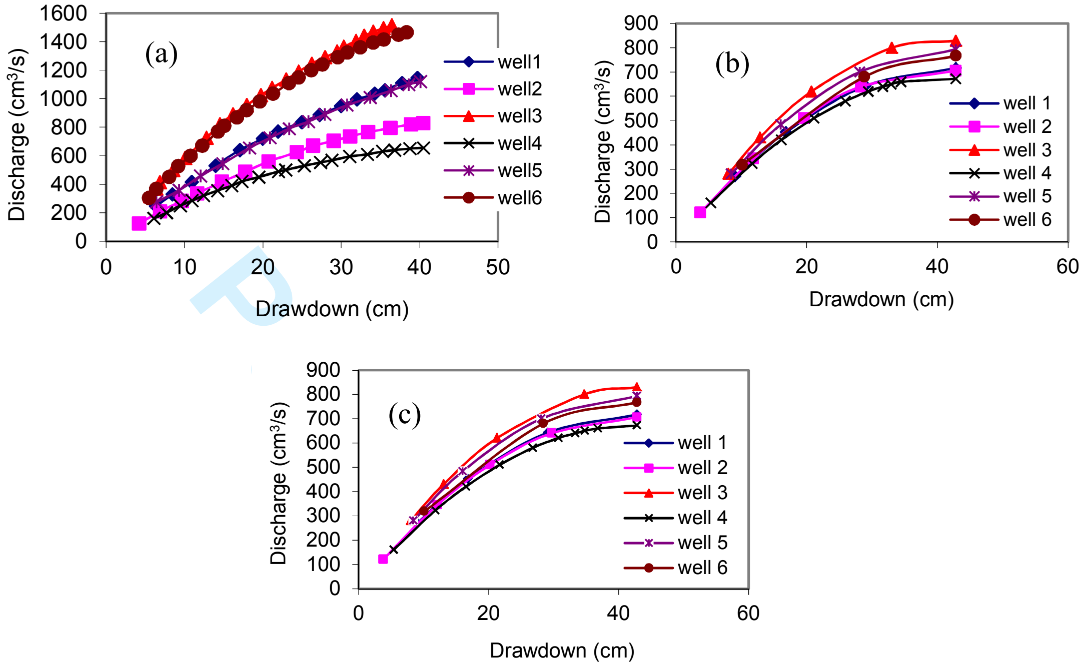

Fig. 5 Discharge-drawdown relationships from (a) Experimental (b) Numerical and (c) Analytical models

This figure shows a considerable agreement between the numerical simulations and the analytical calculations. On the other hand, any of the analytical or the numerical models gives completely different results than the experimental model. It is also seen from Fig. 5 (showing the results of the experimental model of Birpinar and Gazioglu) that well number 5 behaves completely different than the other five wells because its relation is expected to be between those of wells number 6 and 3, but it doesn't appear so. Birpinar and gazioglu (2002) explained in their paper that they repeated the experiments in order to investigate the reasons of such behavior for well number 5 but their results did not change. This is not the case with any of the analytical or the numerical results which give similar behavior as expected for all wells. The pronounced deviations in the experimental results and the different behavior of well number 5 may raise the caution that modeling unconfined flow to a well using smaller scale laboratory models may not be appropriate unless sensitivity analysis to changes in the well radius, model size and other factors are performed. The previous analysis and observations, provided that Darcy's law is applicable, enhances the credibility of each of the analytical and numerical models over the experimental model in order to accurately describe the discharge-drawdown relationship for eccentric wells in a circular island unconfined aquifer under steady state flow conditions. In order to obtain the drawdowns by the analytical and/or the numerical solutions, the discharge rates should not exceed the upper mentioned maximum permissible limits for the corresponding well. For example, the maximum permissible discharge to be used in the analytical and/or the numerical solutions for well number 1 is found to be 716.58 cm^3/sec at which the drawdown reaches 42.8 cm; however, the corresponding discharge rate as given by the experimental model reaches 1180 cm^3/sec for the same drawdown.

6 SUMMARY AND DISCUSSION

A numerical model was developed in order to analyze the discharge-drawdown relationships of eccentric wells operating in circular island unconfined aquifer under steady state flow conditions. The model, uses the finite difference approximation with irregular mesh, included the simulation of a series of one centric and five eccentric wells in the aquifer. For the same problem dimensions, soil properties and flow geometry the results of the present numerical model were compared versus the experimental measurements of Birpinar and Gazioglu (2002) and the analytical calculations using Verruijt equation (1982). From the results of this study the following remarks are of importance:

1. The developed numerical discharge-drawdown relationship for each well very closely matched the analytical calculations using Verruijt equation.

2. Compared to the results of the available experimental model of Birpinar and Gazioglu (2002), the analytical and numerical solutions considerably overestimate the drawdowns for all eccentric wells and slightly underestimate those for the centric well. The maximum percentage difference between the experimental and analytical (or numerical) results is found to be 66% and may be attributed to the fact that in the flow field the seepage velocity of groundwater is highly variable even if the aquifer properties are relatively homogeneous. Thus, in low permeability zones or near stagnation points, the velocity may be very small, however, in high permeability zones or near stress points (such as pumping wells), the velocity may be high. In other words, if the range of validity of Darcy's law which depends on the flow velocity is not satisfied in the experimental model then the comparison

with the analytical (or numerical) models may not be applicable over the entire flow domain and considerable differences may be introduced somewhere in the solution.

3. The experimental model gives completely different discharge-drawdown relationships than any of the analytical equation or the developed numerical model. Moreover, one well in the experimental model (well 5) behaves completely different than expected because its relation should be between those of wells number 6 and 3, but it doesn't appear so. This is not the case with any of the analytical or the numerical results which give similar behaviors as expected for all wells.

4. The pronounced deviations in the experimental results and the different behavior of well number 5 may suggest that the experimental model requires sensitivity analysis to changes in the well radius and model size in addition to the difficulties involved in setting up and carrying out such experiments. On the contrary, numerical models provide an efficient approach in order to study the behavior of eccentric wells in unconfined circular aquifers.

5. Different pumping and/or recharge rates from different wells, generalized well patterns and varying aquifer characteristics can easily be accounted for by the numerical solution.

6. When the well is approximately at mid distance between the center of the aquifer and its boundary (e.g. well 1 and well 2) the percentage difference between the results of the experimental and the analytical (or numerical) models is approximately 20% for small drawdowns (i.e. for drawdowns less than 25% of the aquifer thickness). However, for wells placed near the boundary (at a distance of approximately three quarter the radius of the aquifer, well 3 and well 6) the percentage difference between the results of the experimental and the analytical (or numerical) models reaches 50% for small drawdowns. Moreover, the difference in results increases as the drawdowns increase. For well 4, the maximum discharge that causes maximum drawdown of 42.8 cm has the same magnitude of 672.2 m^3/s, however, the discharges in the experimental model were clearly larger than those required for the analytical (or numerical) models for the same drawdown. The reason of this is due to the extra discharge coming from the seepage face which is neglected in the analytical and the numerical models since they are based on Dupuit's assumption.

7 CONCLUSIONS

On the basis of this comparative analysis, and provided that Darcy's law is applicable, any of the Verruijt analytical equation or the developed numerical model could efficiently be used in order to describe the discharge-drawdown relationship for all values of drawdowns for a well which is placed eccentrically in a circular island unconfined aquifer under steady state flow conditions. Moreover, The founding in this study may be of practical uses in pump and treat systems designed to prevent the contamination from spreading or remove the contaminant mass in unconfined aquifers. This will be achieved by controlling the discharge rate that should be selected as the minimum rate sufficient to prevent enlargement of the contaminated zone or to be much larger than that required for containment so that clean water will flush through the contaminated zone at an expedited rate.

8 REFERENCES

Babbit, H.E. & Caldwell, D.H. (1948) The free surface around and interference between gravity wells. *Univ. of Illinois Bull*, 45(48).

Birpinar, M.E. & Gazioglu S.A. (2002) Discharge-drawdown relationship of eccentric wells. *J. of Hydrolo. Eng.* 7(3): pp. 260-264.

Boulton, N.S. (1951) The flow pattern near a gravity well in a uniform water bearing medium. *J. of the Institute of Civil Eng.*; 36(10): pp. 543-550.

Dupuit, J. (1863) Etudes theoretiques et pratiques sur le mouvement des eauxs dans les canaus de couverts et a travers les terrains permeables, *2nd ed., Dunod*, Paris, pp. 304.

Hantush, M.S. & Jacob, C.E. (1960) Flow to an eccentric well in a leaky circular aquifer. *J. Geophys. Research*, 63(10).

Hantush, M.S. (1964) Hydraulics of wells. *Advances in Hydroscience*; 1, San Diego, CA: pp. 281-432.

Muskat, M. (1937) *The flow of homogeneous fluids through porous media*. McGraw-Hill Book Co.: New York.

Phoolendra K.M. & Kristopher L.K. (2013) Unconfined Aquifer Flow Theory - from Dupuit to present, *arXiv:1304.3987 physics.flu-dyn*, DOI:10.1007/978-1-4614-6479-2_9.

Roger De Wiest, J.M. (1963) Flow to an eccentric well in a leaky circular aquifer with varied lateral replenishment. *Pure and Applied Geophysics*, 54(1): pp. 87-102.

Rushton, K. R. (2003) *Groundwater Hydrology: Conceptual and Computational Models*. John Wiley & Sons Ltd.

Shamsai, A. & Narasimhan, T.N. (1991) Numerical investigation of the free surface-seepage face relationship under steady state conditions. *Water Resour. Res*, 27(3): pp. 409-421.

Siddiraju S. (2013) Pumping Test Analysis Of Large Diameter Wells, *International Journal of Innovative Research & Development*, vol. 2(3), pp. 129-136.

Todd, D.K. (1980) *Groundwater hydrology*. John Wiley and Sons, Inc.: New York.

Verruijt, A. (1982) *Groundwater flow*. Lecture Notes p. 100. Delft,.

Wang H.F. & Anderson M.P. (1982) *Introduction to ground water modeling, finite difference and finite element method.* W. H. Freeman and Company, San Francisco