University of Benghazi
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The General Theory of Banach Algebras

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بسم الله الرحمن الرحيم

قَالُوا سُبْحَانَكَ لاعِلْمَ لَنَا إِلاَ مَاعَلمْتَنَا إِنَكَ أَنتَ الْعَلِيمُ الْحَكِيمُ

صَدَقَ الله العَظِيم

سورة البقرة الآية **32**

Dedication

To my parents.

Acknowledgements

I don't find words articulate enough to express my gratitude for the help and grace that Allah almighty has bestowed upon me .

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Wagdi

Contents

Abstract

In this thesis , we shall discuss the concepts of Banach algebras .

We give some results in the area of Banach algebras .

Also , we discuss the concepts of

- Character mappings on Banach algebras .
- Involution mappings on Banach algebras .
- B*-algebras .

We give some results concerning the previous concepts .

Chapter One

Introduction

In this chapter, we give some standard definitions and results which we shall need later in this thesis .

Notation

Let $\mathbb{R}$ be the set of all real numbers .

Let $\mathbb{C}$ be the set of all complex numbers .

Definition 1.1

Let X be a non-empty set , and let K be the field of scalars ($K = \mathbb{R}$ or $\mathbb{C}$) . Let $x \in X$ and $\alpha \in K$. Then αx is called a *scalar multiplication* .

Definition 1.2

Let X be a non-empty set , and K be the field of scalars ($K = \mathbb{R}$ or $\mathbb{C}$) whose elements are called *vectors* and in which two operations called *addition and scalar multiplication* are defined . Then X is called *a linear space* (or a *vector space*) over K for all $x , y , z \in X$ and $\alpha , \beta \in K$ which satisfies the following conditions :

(i) $x + y = y + x$.

(ii) $(x + y) + z = x + (y + z)$.

(iii) There exists 0 in X such that $x + 0 = x$.

(iv) There exists $-x \in X$ such that $x + (-x) = 0$.

(v) $\alpha (x + y) = \alpha x + \alpha y$.

(vi) $(\alpha + \beta) x = \alpha x + \beta x$.

(vii) $\alpha (\beta x) = (\alpha \beta) x$.

(viii) $1 . x = x$.

Let X be a linear space over K . Then the subtraction is defined by

$$x - y = x + (- y) \quad (x , y \in X).$$

Definition 1.3

Let X be a linear space over K and let $x_1 , x_2 , \ldots , x_n$ be non-zero elements in X . Then $x_1, x_2, \dots, x_n$ are called *linearly independent* if $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0$, then $\alpha_1 = \alpha_2 \dots = \alpha_n = 0$.

Definition 1.4

Let X be a linear space over K . Let $A \subseteq X$. Then A is called a *linear subspace* of X if $\alpha x + \beta y \in A \quad (x , y \in A , \alpha , \beta \in K)$.

Remark

Let A be a linear subspace of a linear space X . Since $0 \in A$, so A is non-empty .

Definition 1.5

An *algebra* is a linear space A over K such that for each ordered pair of elements $x , y \in A$ a unique product $x y \in A$ is defined with the properties

(i) $(x y) z = x (y z)$.

(ii) $x (y + z) = x y + x z$,

$(x + y) z = x z + y z$.

(iii) $\alpha (x y) = (\alpha x) y = x (\alpha y)$,

for all $x , y , z \in A$, $\alpha \in K$.

If $K = \mathbb{R}$ or $\mathbb{C}$, then A is said to be a *real* or *complex algebra* respectively .

Definition 1.6

Let X be a linear space over K , and $E \subseteq X$. Let f , g be mappings of E into X . Let $\alpha \in K$. The natural definition $f + g , \alpha f$ are given by

$$(f + g)(x) = f(x) + g(x) \quad (x \in E),$$

$$(\alpha f)(x) = \alpha f(x).$$

This is called the *pointwise* definition of addition and scalar multiplication. When X is an algebra , the pointwise product is given by

$$(f g)(x) = f(x) g(x) \quad (x \in E).$$

Definition 1.7

Let A be an algebra . We say that A is *commutative* if

$$x\,y = y\,x \quad (x , y \in A) .$$

Otherwise , A is called *non-commutative* .

Definition 1.8

An element e of an algebra A is called an *unit element* or *identity element* if and only if $e \neq 0$ and

$$e\,x = x\,e = x \qquad (x \in A).$$

A unit element e of A is unique .

We say that A is an *algebra with unit* if it has an unit element .

Definition 1.9

Let A be an algebra with unit e . An element $x \in A$ is said to be *invertible* if it has an inverse element in A , that is if A contains an element , written x^{-1} , such that

$$x^{-1}\,x = x\,x^{-1} = e .$$

Then x^{-1} is unique when it exists .

Notation

Let A^{-1} denote the set of all invertible elements of an algebra A .

Theorem 1.1 [10]

Let A and B be complex algebras with the same unit . If $A \subseteq B$,then $A^{-1} \subseteq B^{-1}$.

Theorem 1.2 [10]

Let A be an algebra with unit e . Then

(i) $e^n = e \quad (n \in \mathbb{N})$

(ii) $e^{-1} = e$.

That is , e is an invertible element in A .

Lemma 1.3 [10]

Each non-zero element of $\mathbb{C}$ is invertible .

Theorem 1.4 [10]

Let A be an algebra with unit e . Let x be a non-zero element in A . Then x is invertible in A if and only if x^{-1} is invertible and $(x^{-1})^{-1} = x$.

Theorem 1.5 [10]

Let A be an algebra with unit e . Let x , y be invertible elements of A . Then $x y$ is invertible and

$$(x y)^{-1} = y^{-1} x^{-1} .$$

Lemma 1.6 [10]

Let A be an algebra with unit e . Let x be an invertible element in A . Then αx ($\alpha \neq 0$) is invertible .

Definition 1.10

A subset I of a commutative complex algebra A is said to be an *ideal* if

(i) I is a subspace of A .

(ii) $x y \in I$ whenever $x \in A$ and $y \in I$.

If $I \neq A$, then I is called a *proper ideal* .

Maximal ideals are proper ideals which are not contained in any larger proper ideals .

Definition 1.11

A non-empty subset E of an algebra A is called *subalgebra* of A if

$$x y , y x \in E \quad (x , y \in E).$$

Definition 1.12

Let X and Y be non-empty sets . The *cartesian product of* X and Y is defined by

$$X \times Y = \{ (x , y) : x \in X , y \in Y \}.$$

Note that $X \times Y \neq Y \times X$ unless $X = Y$.

Definition 1.13

Let X be a non-empty set . Let d be a real function defined on the cartesian product $X \times X$ into $\mathbb{R}$ such that for each $x, y, z \in X$

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

Then d is called a *metric* on X and (X, d) is called a *metric space*.

Example 1.1

Let $X = \mathbb{R}$. Define d by

$$d(x, y) = |x - y| \quad (x, y \in X).$$

Then d is a metric on X. This metric space is called the *usual metric space*.

Definition 1.14

Let (X, d) and (Y, d) be metric spaces. A function $f : X \to Y$ is called *continuous at* x_0 in X if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(f(x), f(x_0)) < \varepsilon \quad \text{for all} \quad d(x, x_0) < \delta .$$

The function f is called *continuous on* X if it is continuous at each point of X.

Theorem 1.7 [15]

Let (X, d) be a metric space. Then a distance function d from $X \times X$ into $\mathbb{R}$ is continuous.

Definition 1.15

Let (X, d) be a metric space and $x \in X$. Let $r > 0$. The set

$$B(x, r) = \{ y \in X : d(x, y) < r \}.$$

is called the *open ball* with center x and radius r.

Definition 1.16

Let (X, d) be a metric space. A subset A of X is said to be *open* in X if for each $x \in A$, there is $r > 0$ such that $B(x, r) \subseteq A$.

Definition 1.17

Let (X , d) be a metric space . A subset A of X is said to be *closed* in X if its complement $X - A$ is open in X .

Definition 1.18

Let X be a linear space over K . Let $\|.\| : X \to K$ be a function such that

(i) $\| x \| \geq 0$ for all $x \in X$.

(ii) $\| x \| = 0 \Leftrightarrow x = 0$ for all $x \in X$.

(iii) $\| \alpha x \| = |\alpha| \, \| x \|$ for all $\alpha \in K$, $x \in X$.

(iv) $\| x + y \| \leq \| x \| + \| y \|$ for all $x , y \in X$.

Then $\|.\|$ is called a *norm* on X and $(X , \|.\|)$ is called a *normed space* .

We assume that $\| 1 \| = 1$.

Remark

Let $(X , \|.\|)$ be a normed space . Let $x , y \in X$. Then

(i) $\| x - y \| = \| y - x \|$.

(ii) $\| x \| = \| - x \|$.

Theorem 1.8 [15]

Let $(X , \|.\|)$ be a normed space . Let $x , y \in X$. Then

$$\big| \, \| x \| - \| y \| \, \big| \leq \| x - y \| .$$

Lemma 1.9 [15]

Every normed space $(X , \|.\|)$ is a metric space with the distance

$$d (x , y) = \| x - y \| \quad (x , y \in X) .$$

Remark

In general , the converse of Lemma 1.9 is not true .

For example :

Let $X = \mathbb{R}$.

Let d_1 be a metric on X .

Define d_2 by

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)} \quad (x, y \in X).$$

Then d_2 is a metric on X but d_2 is not a norm on X because

$$d_2(\alpha x, \alpha y) \neq \alpha d_2(x, y) .$$

Lemma 1.10 [15]

Let $(X, \|.\|)$ be a normed space . Then a norm function is continuous .

Definition 1.19

Let X, Y be linear spaces over K . A function $f : X \to Y$ is called *linear* if

(i) $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

(ii) $f(\alpha x) = \alpha f(x)$ for all $\alpha \in K, x \in X$,

or , f is *linear* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad (x, y \in X, \alpha, \beta \in K) .$$

Lemma 1.11 [15]

Let X, Y be linear spaces over K . Let $f : X \to Y$ be a linear function . Then $f(0) = 0$.

Remark

In general , the converse of Lemma 1.11 is not true .

For example :

Define f by

$$f(x) = x^2 .$$

Then $f(0) = 0$ but f is not linear .

Definition 1.20

Let $(X, \|.\|)$ be a normed space . A function f on X is called *bounded* if there exists a positive integer M such that

$$\| f(x) \| \leq M \quad \text{for all } x \in X .$$

If f is a linear map , then

$$\| f(x) \| \leq \| f \| \, \| x \| \quad \text{for all } x \in X ,$$

or

$$\| f(x) \| \leq M \|x\| \quad \text{for all} \quad x \in X .$$

Definition 1.21

A linear functional on a linear space X over K is a linear function from X into K.

Definition 1.22

Let $(X, \|.\|)$ be a normed space. A linear functional $f : X \to K$ is called *bounded* if there exists a positive integer M such that

$$| f(x) | \leq M \|x\| \qquad (x \in X).$$

Theorem 1.12 [16]

Let $(X, \|.\|)$ be a normed space. A linear functional on X is continuous if and only if it is bounded.

Theorem 1.13 [16]

Let f be a bounded linear functional (or continuous linear functional) on a normed space X. If $x_0 \in X$ such that $f(x_0) = 0$, then $x_0 = 0$.

Definition 1.23

Let A, B be complex algebras over K. A mapping f of A into B is called *homomorphism* if f is linear and

$$f(xy) = f(x) f(y) \qquad (x, y \in A).$$

Definition 1.24

Let A, B be complex algebras over K. A one-one homomorphism mapping from A onto B is called *isomorphism*.

Definition 1.25

A function f is said to be *analytic* on the domain D of $\mathbb{C}$ if it has derivative at each point of D. Then f is called an *entire function* if it is analytic at each point of $\mathbb{C}$.

Theorem 1.14 (Leibnitz′s Rule) [14]

$$(fg)^{(n)} = \sum_{j=0}^{n} \binom{n}{j} f^{(j)} g^{(n-j)} \qquad (n = 1, 2, ...),$$

where f and g are n-times continuously differentiable functions .

Theorem 1.15 (Liouville) [14]

If f is bounded and entire function on the complex plane , then f is constant .

Definition 1.26

Let A be a subset of $\mathbb{R}$. An element $x \in \mathbb{R}$ is called an *upper bound* of A if $a \leq x$ for all $a \in A$.

If A has an upper bound , then A is called *bounded above set* .

Definition 1.27

Let A be a subset of $\mathbb{R}$. An element $y \in \mathbb{R}$ is called a *lower bound* of A if $y \leq a$ for all $a \in A$.

If A has a lower bound , then A is called *bounded below set* .

Definition 1.28

Let A be a subset of $\mathbb{R}$ .Then A is called *bounded* if it is both bounded above and bounded below .

Definition 1.29

Let A be a subset of $\mathbb{R}$. A real number u is called a *supremum* of A

(The least upper bound of A) if

(i) u is an upper bound of A .

(ii) If v be any upper bound of A . Then $u \leq v$.

It is denoted by $\sup(A)$.

Theorem 1.16 [2]

Let A be a non-empty bounded above subset of $\mathbb{R}$. Then A has a supremum and it is unique .

Definition 1.30

Let A be a subset of $\mathbb{R}$. A real number w is called an *infimum* of A

(The greatest lower bound of A) if

(i) w is an lower bound of A .

(ii) If t be any lower bound of A . Then $t \leq w$.

It is denoted by $\inf(A)$.

Theorem 1.17 [2]

Let A be a non-empty bounded below subset of $\mathbb{R}$ *. Then A has an infimum and it is unique .*

Theorem 1.18 [2]

Let A be a non-empty bounded subset of $\mathbb{R}$ *. Then A has a supremum and an infimum .*

Theorem 1.19 [8]

Let A be a bounded set of real numbers and let $\varepsilon > 0$ *. Then* $a = \inf(A)$ *if and only if there exists at least* $x \in A$ *such that* $x < a + \varepsilon$.

Theorem 1.20 [8]

Let A be a bounded set and $B \subset A$ *. Then B is also bounded .*

Notation

Let $C[a,b]$ be the space of all complex- valued continuous functions on $[a,b]$.

Theorem 1.21 [8]

If $f \in C[a,b]$*, and if* $M = \sup_{a \le x \le b} |f(x)|$ *, then there is* $a \le x_0 \le b$ *, such that* $|f(x_0)| = M$.

Theorem 1.22 [8]

Let X be a bounded set of $\mathbb{R}$ *and let* $f : X \to \mathbb{R}$ *be a bounded function . Then*

(i) $\sup_{x \in X} (\alpha f(x)) = \alpha \sup_{x \in X} (f(x)) \quad (\alpha > 0)$.

(ii) $\sup_{x \in X} (\alpha f(x)) = \alpha \inf_{x \in X} (f(x)) \quad (\alpha < 0)$.

Definition 1.31

Let X be a non-empty set and let T be a collection of subsets of X such that

(i) $X, \varnothing \in T$.

(ii) If $O_1, O_2 \in T$, then $O_1 \cap O_2 \in T$.

(iii) If for each $\alpha \in I$, $O_\alpha \in T$, then $\bigcup_{\alpha \in I} O_\alpha \in T$.

Then T is called a *topology* on X and (X, T) is called a *topological space*. The members of T are called *open sets*.

Definition 1.32

Let (X, T) be a topological space and $A \subset X$. A point $x \in A$ is an *interior* point of A if there exists an open set O such that $x \in O \subset A$. The set of all interior points of A is denoted by $\operatorname{int}(A)$.

Definition 1.33

Let (X, T) be a topological space and $x \in X$. Let A be a subset of X. Then x is called *a boundary point* of A if for every open set O containing x, then $O \cap A \neq \varnothing$ and $O \cap (X \setminus A) \neq \varnothing$.

The set of all boundary points of A is denoted by $\partial(A)$.

Theorem 1.23 [17]

Let (X, T) be a topological space. Then A is open if and only if $\partial(X) \cap A = \varnothing$.

Definition 1.34

Let (X, T) be a topological space and $x \in X$. Let $A \subset X$. Then x is called a *closure point* of A if for every open set O containing x, then

$$O \cap A \neq \varnothing .$$

The set of all closure points of A is denoted by $\overline{A}$.

Theorem 1.24 [17]

Let (X, T) be a topological space and $A \subset X$. Then

(i) $A \subset \overline{A}$

(ii) A *is closed if and only if* $A = \overline{A}$

(iii) $\overline{A}$ *is the smallest closed set containing* A.

Definition 1.35

Let X and Y be topological spaces and let f be a function from X into Y. Then f is called *homeomorphism* if

(i) f is one-one and onto.

(ii) f and f^{-1} are continuous.

Definition 1.36

Let (X, T) be a topological space. A collection $\{u_\alpha\}_{\alpha \in I}$ of open sets is called an *open cover* of X if $X = \bigcup_{\alpha \in I} u_\alpha$.

A collection $\{u_{\alpha_i}\}_i$ of a topological space (X, T) is called an *open subcover* of $\{u_\alpha\}_{\alpha \in I}$ if

$$\{u_{\alpha_i}\}_i \subseteq \{u_\alpha\}_\alpha, \text{ and } X = \bigcup_i u_{\alpha_i}.$$

Definition 1.37

A topological space (X, T) is said to be *compact* if each open cover of X has a finite open subcover.

Theorem 1.25 [17]

A closed subset A of a compact space X is compact.

Theorem 1.26 (Heine - Borel) [17]

A subset A of $\mathbb{R}$ is compact if and only if A is closed and bounded.

Definition 1.38

A topological space (X, T) is called *Hausdorff* if every distinct points $x, y \in X$, there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Definition 1.39

A sequence (a_n) in a metric space (X, d) is called *convergent* to a point a in X if for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(a_n, a) < \varepsilon \qquad (n > N).$$

In a normed space $(X, \|.\|)$,

$$\| a_n - a \| < \varepsilon \qquad (n > N).$$

Theorem 1.27 [15]

Let $(X, \|.\|)$ be a normed space. If $x_n \to x$ $(n \to \infty)$ in X, then

$$\| x_n \| \to \| x \| \text{ in } \mathbb{R}.$$

Definition 1.40

The sequence (a_n) is said to *tend to infinity* if given A (*however large*), there exists N such that

$$a_n > A \quad \text{for all} \quad n > N .$$

We use the arrow notation and we write $a_n \to \infty$.

Definition 1.41

Let (X, d) and (Y, d) be two metric spaces . Let (x_n) be a sequence in (X, d) . A function $f : (X, d) \to (Y, d)$ is called *continuous* at x_0 in X if $x_n \to x_0$ *in* X , then $f(x_n) \to f(x_0)$.

Definition 1.42

A sequence (a_n) in a metric space (X, d) is called *cauchy* in X if for each $\varepsilon > 0$, there exists a positive integer N such that

$$d(a_n, a_m) < \varepsilon \qquad (n, m > N).$$

In a normed space $(X, \|.\|)$,

$$\| a_n - a_m \| < \varepsilon \qquad (n, m > N).$$

Theorem 1.28 [2]

Every convergent sequence is a Cauchy sequence .

Remark

In general , the converse of Theorem 1.28 is not true .

For example :

Let $X = \mathbb{R} \setminus \{0\}$.

Let $a_n = \dfrac{1}{n} \quad (n \in \mathbb{N})$.

Then (a_n) is a Cauchy sequence in X , but (a_n) does not converge in X .

Definition 1.43

Let $(X, \|.\|)$ be a normed space . A sequence (a_n) on X is called *bounded* if there exists a positive integer M such that

$$\| a_n \| \leq M \qquad (n \in \mathbb{N}) .$$

Lemma 1.29 [2]

Let $(X , \|.\|)$ *be a normed space . If* $a_n \to 0 \; (n \to \infty)$ *in* X *and* (b_n) *is a bounded sequence , then* $(a_n \, b_n) \to 0$ *in* X .

Theorem 1.30 [2]

Every convergent sequence is bounded .

Remark

In general , the converse of Theorem 1.30 is not true .

For example :

Let $a_n = (-1)^n \quad (n \in \mathbb{N})$.

Then (a_n) is a bounded sequence but not convergent .

Theorem 1.31 [2]

Every Cauchy sequence is bounded .

Remark

In general , the converse of Theorem 1.31 is not true .

For example :

Let $a_n = (-1)^n \quad (n \in \mathbb{N})$.

Then (a_n) is a bounded sequence but not Cauchy .

Definition 1.44

A metric space (X , d) is called *complete* if every Cauchy sequence in (X , d) is convergent in (X , d) .

Definition 1.45

A complete normed space $(X , \|.\|)$ is called a *Banach space* .

We state some examples concerning Banach spaces .

Examples 1.2 [4 , 9]

(i) Let $\mathbb{R}$ be the algebra of real numbers .We define $\mathbb{C}$ as $\mathbb{R} \times \mathbb{R}$, ($\mathbb{C}$ is the set of all complex numbers) , with operations given by

$$(a , b) + (c , d) = (a + c , b + d).$$

$$\alpha (a , b) = (\alpha a , \alpha b)$$

$$(a , b) \; (c , d) = (a c - b d , a d + b c)$$

The norm on $\mathbb{R}$ is given by

$$\| x \| = | x | \quad (x \in \mathbb{R}).$$

Also, the norm on $\mathbb{C}$ is given by

$$\| x \| = | x | \quad (x \in \mathbb{C}).$$

Then $\mathbb{R}$ and $\mathbb{C}$ are Banach spaces.

(ii) Let $M_{n \times n}$ denote the set of all $n \times n$ matrices $A = (a_{ij})$ with complex entries a_{ij}.

The addition on $M_{n \times n}$ is given by

$$A = (a_{ij}) \ , \ B = (b_{ij})$$

$$A + B = (a_{ij} + b_{ij})$$

The scalar multiplication is given by

$$\alpha A = \alpha (a_{ij}) = (\alpha a_{ij}).$$

The usual matrix multiplication is given by

$$(A B)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} .$$

The norm on $M_{n \times n}$ is defined by

$$\| A \| = \max \left\{ \sum_{j=1}^{n} | a_{ij} | : 1 \le i \le n \right\} .$$

Then $M_{n \times n}$ is a Banach space.

(iii) Let $C [a , b]$ be the space of all complex- valued continuous functions on $[a , b]$.

With the pointwise addition, scalar multiplication and pointwise product and with the norm is given by

$$\| f \| = \sup_{x \in X} (| f (x) |), \ (f \in C [a , b]) ,$$

is a Banach space.

(iv) Let $C^{n} [a , b]$ be the space of all complex-valued functions on $[a , b]$ which are n-times continuously differentiable.

With the pointwise addition, scalar multiplication and pointwise product and with the norm is given by

$$\| f \| = \sum_{k=0}^{n} \frac{1}{k!} \| f^{(k)} \|_{\infty} \quad (f \in C^{n}[a,b]),$$

where $\| f \|_{\infty} = \sup_{a \le x \le b} | f(x) |$,

is a Banach space.

(v) Let $X \neq \{0\}$. Let $B(X, X)$ be the space of all bounded linear mappings from a normed space $(X, \|.\|)$ into itself.

With the pointwise addition, scalar multiplication and the multiplication of $T_1, T_2 \in B(X, X)$ as a composition of operator:

$$(T_1 T_2)(x) = T_1(T_2(x)), (x \in X),$$

and the norm is given by

$$\| T \| = \sup \{ \| T(x) \| : \| x \| \le 1 \}, (T \in B(X, X)),$$

is a Banach space.

(vi) Let $\mathcal{D} = \{ z \in \mathbb{C} : | z | \le 1 \}$.

Then $\mathcal{D}$ is called a *unit disc* in $\mathbb{C}$.

$$\operatorname{int}(\mathcal{D}) = \{ z \in \mathbb{C} : | z | < 1 \}.$$

Let $A(\mathcal{D})$ denote the family of all continuous functions on $\mathcal{D}$ and analytic functions on $\operatorname{int}(\mathcal{D})$. Then $A(\mathcal{D})$ is called the *disc algebra*.

With the pointwise addition, scalar multiplication and pointwise product and the norm is given by

$$\| f \| = \sup_{z \in \mathcal{D}} (| f(z) |), \quad (f \in A(\mathcal{D})),$$

is a Banach space.

(vii) Let $L^1(\mathbb{R})$ denote the space of integrable complex valued functions on $\mathbb{R}$. That is

$$L^1(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \| f \| = \int_{-\infty}^{\infty} | f(x) | \, dx < \infty \right\}.$$

With the pointwise addition, scalar multiplication and with the norm is given by

$$\| f \| = \int_{-\infty}^{\infty} | f(x) | \, dx \qquad (x \in \mathbb{R}),$$

is a Banach space with

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x) \, g(t - x) \, dx$$

as the product .

(viii) $\ell^1(\mathbb{Z}) = \{ a = (a_n : n \in \mathbb{Z}) : \sum_{n=-\infty}^{\infty} | a_n | < \infty \},$

where $\mathbb{Z}$ is the set of all integers . With the pointwise addition , scalar multiplication , the product of ℓ^1 is given by

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k} \, b_k \quad (n \in \mathbb{Z}),$$

and with The norm on ℓ^1 is given by

$$\| a \| = \sum_{n=-\infty}^{\infty} | a_n | \ ,$$

is a Banach space .

(ix) Let A be a normed space over K . Let $A^{\#}$ be the set of all ordered pairs (x, λ) , where $x \in A$ and $\lambda \in \mathbb{C}$.

The addition , scalar multiplication and the product defined for all $x, y \in A$ and $\lambda_1, \lambda_2 \in K$ by

$$(x, \lambda_1) + (y, \lambda_2) = (x + y, \lambda_1 + \lambda_2),$$

$$\lambda_2 (x, \lambda_1) = (\lambda_2 x, \lambda_1 \lambda_2)$$

$$(x, \lambda_1)(y, \lambda_2) = (xy + \lambda_1 y + \lambda_2 x, \lambda_1 \lambda_2) .$$

Let $x \in A$. Then $x \to (x, 0)$ and

$$(x, \lambda) = (x, 0) + \lambda (0, 1) \quad (\lambda \in \mathbb{C}).$$

The norm on $A^{\#}$ is defined by

$$\| (x, \alpha) \| = \| x \| + | \alpha | \quad (x \in A, \alpha \in \mathbb{C}),$$

where $\| x \|$ is a norm on A .

The identity element of $A^{\#}$ is $\tilde{e} = (0,1)$, and

$$\| (0,1) \| = 0 + |1| = 1.$$

If A is a Banach space, then $A^{\#}$ is a Banach space.

Definition 1.46

Let E be a linear space over K. Let $B : E \times E \to K$ such that

(i) $B(x, x) \geq 0 \quad (x \in E)$.

(ii) $B(x, x) = 0 \iff x = 0$.

(iii) $B(\alpha x + \beta y, z) = \alpha B(x, z) + \beta B(y, z)$

$$(x, y, z \in E, \ \alpha, \beta \in K).$$

(iv) $B(x, y) = \overline{B(y, x)} \quad (x, y \in E)$.

Then B is called an *inner product* on E.

Definition 1.47

A *Hilbert space* H is a Banach space in which the norm is defined by inner product

$$\| x \| = \sqrt{B(x, x)} \quad (x \in H),$$

and we write $B(x, y) = \langle x, y \rangle$.

Definition 1.48

Let T be a bounded linear mapping on H. The unique bounded linear mapping T^* on H that satisfies

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad (x, y \in H),$$

is called the *Hilbert space adjoint* of T.

Notation

Let $BL(H)$ denote the set of all bounded linear mappings on H.

Theorem 1.32 [4]

Let $T, S \in BL(H)$. *Then*

(i) $(T + S)^* = T^* + S^*$

(ii) $(\alpha T)^* = \overline{\alpha} \, T^* \quad (\alpha \in \mathbb{C})$.

(iii) $(TS)^* = S^* T^*$

(iv) $(T^{*})^{*} = T$.

(v) $I^{*} = I$, I is the identity mapping .

Theorem 1.33 [4]

Let $T \in BL(H)$. *Then*

$$\| T T^{*} \| = \| T^{*} T \| = \| T \|^{2} .$$

Chapter Two
Banach algebras

2.1 Banach algebras

Banach algebras were introduced in 1940 by the Russian mathematical I . M . Gelfand .

Definition 2.1.1

A *normed algebra* A is an algebra which is a normed space (A , $\|\cdot\|$) and in which

$$\| x\, y \| \le \| x \| \, \| y \| \qquad (x , y \in A).$$

We shall state and prove some results concerning normed algebras .

Lemma 2.1.1

Let A be a normed algebra with unit e . Then $\| e \| \ge 1$.

Proof

Let $x \in A$ with $x \ne 0$. Then

$$x\, e = e\, x = x \, .$$

So

$$\| x\, e \| = \| x \| \, .$$

We obtain

$$\| x\, e \| \le \| x \| \, \| e \| .$$

Therefore

$$\| x \| \le \| x \| \, \| e \| ,$$

and so

$$\| e \| \ge 1 \, .$$

Similarly , if $e\, x = x$, then $\| e \| \ge 1$.

Remark

We shall make the additional assumption that $\| e \| = 1$.

Lemma 2.1.2

Let A be a normed algebra. Let $x \in A$, $n \in \mathbb{N}$. Then

$$\| x^{n} \| \leq \| x \|^{n} .$$

Proof

We use mathematical induction

Let $n = 1$. Then

$$\| x \| \leq \| x \| \qquad (x \in A).$$

Now, suppose it is true for $n = k$

$$\| x^{k} \| \leq \| x \|^{k} \qquad (x \in A).$$

Now, we shall prove that it is true for $n = k + 1$.

We have

$$\| x^{k+1} \| = \| x^{k} x \| \qquad (x \in A).$$

$$\leq \| x^{k} \| \, \| x \|$$

$$\leq \| x \|^{k} \| x \|$$

$$= \| x \|^{k+1} .$$

Thus $\| x^{k+1} \| \leq \| x \|^{k+1}$.

Hence $\| x^{n} \| \leq \| x \|^{n}$.

Lemma 2.1.3

Let A be a normed algebra. Let $x \in A$ and $n, m \in \mathbb{N}$. Then

$$\| x^{n+m} \| \leq \| x \|^{n+m} .$$

Proof

Let $x \in A$. Then

$$\| x^{n+m} \| = \| x^{n} x^{m} \|$$

$$\leq \| x^n \| \, \| x^m \|$$

$$\leq \| x \|^n \| x \|^m \qquad (\text{ Lemma 2.1.2 })$$

$$= \| x \|^{n+m} .$$

Theorem 2.1.4

Let A be a normed algebra. If $x_n \to x$, $y_n \to y$ $(n \to \infty)$ in A , then $x_n y_n \to x y$.

Proof

Let $x_n \to x$ and $y_n \to y$ in A . Then

$$\| x_n y_n - x y \| = \| x_n y_n - x_n y + x_n y - x y \|$$

$$= \| x_n (y_n - y) + y (x_n - x) \|$$

$$\leq \| x_n (y_n - y) \| + \| y (x_n - x) \|$$

$$\leq \| x_n \| \| y_n - y \| + \| y \| \| x_n - x \|$$

$$\to 0 \quad (n \to \infty) .$$

Hence $x_n y_n \to x y$.

Theorem 2.1.5

Let (x_n) and (y_n) be bounded sequences in a normed algebra A . Then $(x_n y_n)$ is a bounded sequence in A .

Proof

Let (x_n) be a bounded sequence in A . Then there exists a positive integer M_1 such that

$$\| x_n \| \leq M_1 \quad \text{for all } n .$$

Let (y_n) be a bounded sequence in A . Then there exists a positive integer M_2 such that

$$\| y_n \| \le M_2 \quad \text{for all} \quad n .$$

We have

$$\| x_n y_n \| \le \| x_n \| \, \| y_n \|$$
$$\le M_1 M_2 .$$

Choose $M = M_1 M_2 > 0$.

It follows that

$$\| x_n y_n \| \le M \quad \text{for all} \quad n .$$

Hence $(x_n y_n)$ is a bounded sequence .

Theorem 2.1.6

Let A be a normed algebra . If (x_n) and (y_n) are Cauchy sequences in A, then $(x_n y_n)$ is a Cauchy sequence in A .

Proof

Since (x_n) is a Cauchy sequence in A , so (x_n) is a bounded sequence (Theorem 1.31) . Then there exists a positive integer M such that

$$\| x_n \| \le M \qquad (n \in \mathbb{N}).$$

For each $\varepsilon > 0$, there exists a positive integer N such that

$$\| x_n - x_m \| < \frac{\varepsilon}{2M} \quad (n , m > N).$$

Also , since (y_n) is a Cauchy sequence , so (y_n) is a bounded sequence . Then there exists a positive integer M such that

$$\| y_n \| \le M \qquad (n \in \mathbb{N}).$$

Similarly , for each $\varepsilon > 0$, there exists a positive integer N such that

$$\| y_n - y_m \| < \frac{\varepsilon}{2M} \quad (n , m > N) .$$

We have

$$\| x_n y_n - x_m y_m \| = \| x_n y_n - x_m y_n + x_m y_n - x_m y_m \|$$

$$= \| y_n (x_n - x_m) + x_m (y_n - y_m) \|$$

$$\leq \| y_n (x_n - x_m) \| + \| x_m (y_n - y_m) \|$$

$$\leq \| y_n \| \| x_n - x_m \| + \| x_m \| \| y_n - y_m \|$$

$$\leq M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon .$$

Hence $(x_n y_n)$ is a Cauchy sequence in A.

Definition 2.1.2

Let $(A, \| . \|)$ be a normed algebra. If A is complete with relative to this norm (i.e, A is a Banach space), then A is called *a Banach algebra*. We give some examples concerning Banach algebras.

Examples 2.1

(i) The space $\mathbb{R}$ is a Banach space with the norm

$$\| x \| = | x | \qquad (x \in \mathbb{R}).$$

Let $x, y \in \mathbb{R}$. Then

$$\| x y \| = | x y |$$

$$= | x | | y |$$

$$= \| x \| \| y \|.$$

Hence $\mathbb{R}$ is a normed algebra. Then $\mathbb{R}$ with the usual addition and scalar multiplication and pointwise product is a commutative Banach algebra.

Also, $\mathbb{C}$ with the usual structure and the norm

$$\| x \| = | x | \qquad (x \in \mathbb{C}),$$

is a commutative Banach algebra.

(ii) The norm on $M_{n \times n}$ is given by

$$\| A \| = \max \left\{ \sum_{j=1}^{n} | a_{ij} | : 1 \leq i \leq n \right\} \quad (A \in M_{n \times n}).$$

Then $M_{n \times n}$ is a Banach space.

Let $A=(a_{ij}), B=(b_{ij})$. Let $A, B \in M_{n\times n}$. Then

$$\| AB \| \le \| A \| \, \| B \| .$$

Hence $M_{n\times n}$ is a Banach algebra. As is well-known matrix multiplication is not commutative.

(iii) The norm on $C[a,b]$ is given by

$$\| f \| = \sup_{a\le x\le b} (|f(x)|) \quad (f \in C[a,b]).$$

Then $C[a,b]$ is a Banach space.

Let $f, g \in C[a,b]$. Then

$$\| fg \| = \sup_{a\le x\le b} (|f(x)\, g(x)|).$$

By Theorem 1.21, there exsists x_0 in $[a,b]$ such that

$$\| fg \| = |f(x_0)|\, |g(x_0)|$$

$$\le \| f \| \, \| g \|.$$

Hence $C[a,b]$ is a commutative Banach algebra.

(iv) The norm on $C^n[a,b]$ is given by

$$\| f \| = \sum_{k=0}^{n} \frac{1}{k!} \| f^{(k)} \|_\infty \quad (f \in C^n[a,b]).$$

Then $C^n[a,b]$ is a Banach space.

Let $f, g \in C^n[a,b]$. Then

$$\| fg \| = \sum_{k=0}^{n} \frac{1}{k!} \left\| (fg)^{(k)} \right\|_\infty$$

$$= \sum_{k=0}^{n} \frac{1}{k!} \left\| \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)} \right\|_\infty$$

$$= \sum_{k=0}^{n} \left\| \sum_{j=0}^{k} \frac{1}{j!\,(k-j)!} f^{(j)} g^{(k-j)} \right\|_\infty$$

$$\leq \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{1}{j!} \left\| f^{(j)} \right\|_{\infty} \frac{1}{(k-j)!} \left\| g^{(k-j)} \right\|_{\infty}$$

$$\leq \sum_{l=0}^{n} \sum_{j=0}^{n} \frac{1}{j!} \left\| f^{(j)} \right\|_{\infty} \frac{1}{l!} \left\| g^{(l)} \right\|_{\infty}$$

$$= \| f \| \| g \| .$$

Thus $\| f g \| \leq \| f \| \| g \|$.

Hence $C^n[a,b]$ is a commutative Banach algebra .

(v) The norm on $B(X,X)$ is given by

$$\| T \| = \sup \{ \| T(x) \| : \| x \| \leq 1 \} \quad (T \in B(X,X)).$$

Then $B(X,X)$ is a Banach space .

Let $T_1, T_2 \in B(X,X)$. Then

$$\| (T_1 T_2)(x) \| = \| T_1 (T_2(x)) \|$$

$$\leq \| T_1 \| \| T_2(x) \|$$

$$\leq \| T_1 \| \| T_2 \| \| x \|.$$

Thus

$$\| T_1 T_2 \| \leq \| T_1 \| \| T_2 \|.$$

Hence $B(X,X)$ is a Banach algebra .

(vi) Let $A(\mathcal{D})$ be the disc algebra with the norm

$$\| f \| = \sup_{z \in \mathcal{D}} (| f(z) |) \quad (f \in A(\mathcal{D})).$$

Then $A(\mathcal{D})$ is a Banach space .

Let $f, g \in A(\mathcal{D})$. Then

$$\| f g \| \leq \| f \| \| g \|.$$

Hence $A(\mathcal{D})$ is a commutative Banach algebra .

(vii) The norm on $L^1(\mathbb{R})$ is

$$\| f \| = \int_{-\infty}^{\infty} | f(x) | \, dx \qquad \big(x \in \mathbb{R} , f \in L^1(\mathbb{R}) \big).$$

Then $L^1(\mathbb{R})$ is a Banach space and the product is given by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x) g(t - x) \, dx .$$

Let $f, g \in L^1(\mathbb{R})$. Then

$$\| f * g \| = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(x) g(t - x) | \, dt \, dx$$

$$= \int_{-\infty}^{\infty} | f(x) | \Big(\int_{-\infty}^{\infty} | g(t - x) | \, dt \Big) dx$$

$$= \int_{-\infty}^{\infty} | f(x) | \, \| g \| \, dx$$

$$= \| f \| \, \| g \| .$$

Hence $L^1(\mathbb{R})$ is a commutative Banach algebra .

(viii) The norm on ℓ^1 is given by

$$\| a \| = \sum_{n=-\infty}^{\infty} | a_n | \qquad (a \in \ell^1) .$$

Then ℓ^1 is a Banach space and the product is given by

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k} \, b_k \quad (n \in \mathbb{Z}).$$

Let $a, b \in \ell^1$. Then

$$\sum_{n \in \mathbb{Z}} | (a * b)_n | = \sum_{n \in \mathbb{Z}} | \sum_{k \in \mathbb{Z}} a_{n-k} \, b_k |$$

$$\leq \sum_{n \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} | a_{n-k} | \, | b_k |$$

$$= \Big(\sum_{k \in \mathbb{Z}} | b_k | \Big) \Big(\sum_{n \in \mathbb{Z}} | a_{n-k} | \Big)$$

$$= \| b \| \, \| a \|.$$

Hence ℓ^1 is a normed algebra .Thus ℓ^1 is a Banach algebra .

$$(a * b)_n = \sum_{k \in \mathbb{Z}} a_{n-k} \; b_k$$

$$= \sum_{k \in \mathbb{Z}} b_k \; a_{n-k} \; .$$

Set $u = n - k$. Then

$$(a * b)_n = \sum_{n-u \in \mathbb{Z}} b_{n-u} \; a_u \; .$$

Hence ℓ^1 is commutative .

(ix) Let A be a normed space over K . Let $A^{\#}$ be the set of all ordered pairs (x, λ) , where $x \in A$ and $\lambda \in \mathbb{C}$.

The norm on $A^{\#}$ is given by

$$\|(x, \lambda)\| = \|x\| + |\lambda| \, .$$

Then $A^{\#}$ is a Banach space .

Let A be a normed algebra . Let $(x_1, \lambda_1), (x_2, \lambda_2) \in A^{\#}$. Then

$$\|(x_1, \lambda_1)(x_2, \lambda_2)\| = \|(x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1 \lambda_2)\|$$

$$= \|x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1\| + |\lambda_1 \lambda_2|$$

$$\leq \|x_1 x_2\| + \|\lambda_1 x_2\| + \|\lambda_2 x_1\| + |\lambda_1 \lambda_2|$$

$$\leq \|x_1\| \|x_2\| + |\lambda_1| \|x_2\| + |\lambda_2| \|x_1\| + |\lambda_1| |\lambda_2|$$

$$= (\|x_1\| + |\lambda_1|)(\|x_2\| + |\lambda_2|)$$

$$= \|(x_1, \lambda_1)\| \; \|(x_2, \lambda_2)\|.$$

Thus $A^{\#}$ is a normed algebra .

Hence $A^{\#}$ is a Banach algebra with unit $\tilde{e} = (0,1)$.

If A is commutative , then $A^{\#}$ is commutative .

Definition 2.1.3

Let X be a compact Hausdorff space . Let A be a subset of $C(X)$. Then A is called *separates the points* of X , if for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Definition 2.1.4

Let A be a subset of $C(X)$. Then A is called *self-adjoint* if $f \in A$, then $\overline{f} \in A$.

Theorem 2.1.7 (Stone-Weierstrass) [16]

Let X be a compact Hausdorff space . Let A be a subalgebra of $C(X)$ and separating the points of X . If A is self-adjoint , then

$$\overline{A} = C(X).$$

Remark

There are some Banach algebras which are not closed .
For example :

Let $A = C^1[0,1]$.

Then A is a Banach algebra (Example 2.1 (iv)) .
By Stone-Weierstrass theorem , we obtain

$$\overline{C^1[0,1]} = C[0,1].$$

It follows that $C^1[0,1]$ is not closed .

Theorem 2.1.8 [9]

Let A be a complex Banach algebra with unit . Then every closed subalgebra of A is itself a Banach algebra .

Theorem 2.1.9

Let A be a complex Banach algebra and suppose x in A is such that $\| x \| < 1$. Then there exists $y \in A$ such that $xy = x + y$.

Proof

Since $\| x \| < 1$ and $\| x^n \| \leq \| x \|^n$, the series $-x - x^2 - x^3 - \ldots$ is absolutely convergent . Since A is a Banach space , so the series converges.

Let the sum of the series be y . Then

$$xy = -x^2 - x^3 - x^4 - \ldots$$
$$= x + y .$$

Theorem 2.1.10 [14]

Let A be a commutative Banach algebra with unit . Then every maximal ideal of A is closed .

Theorem 2.1.11 [14]

Let A be a complex Banach algebra with unit . Let I be an ideal of A . Then the closure of I is an ideal .

2.2 Invertible elements of Banach algebras

Theorem 2.2.1 [9]

Let A be a complex Banach algebra with unit e . If $x \in A$ satisfies $\| x \| < 1$, then $e - x$ is invertible , and

$$(e - x)^{-1} = e + \sum_{n=1}^{\infty} x^n .$$

Theorem 2.2.2 [9]

Let A be a complex Banach algebra with unit e . If $x \in A$ and $\| x \| < 1$, then $e + x$ is invertible , $(e + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, and

$$\| (e + x)^{-1} - e + x \| \leq \frac{\| x \|^2}{1 - \| x \|} .$$

Theorem 2.2.3 [9]

Let A be a complex Banach algebra with unit e . If $\| x - e \| < 1$, then x is invertible and

$$x^{-1} = e + \sum_{n=1}^{\infty} (e - x)^n .$$

Theorem 2.2.4

Let A be a complex Banach algebra with unit e . Then A^{-1} is an open subset of A .

Proof

Let $x_0 \in A^{-1}$. Let $B(x_0, \varepsilon)$ be an open ball with center x_0 and radius ε.

Set $\varepsilon = \dfrac{1}{\| x_0^{-1} \|} > 0$.

We will show that $B(x_0, \varepsilon) \subseteq A^{-1}$. Let $x \in B(x_0, \varepsilon)$. Then

$$\| x - x_0 \| < \frac{1}{\| x_0^{-1} \|} .$$

Let $y = x_0^{-1} x$ and $z = e - y$. Then

$$\begin{aligned} \| z \| &= \| - z \| \\ &= \| y - e \| \\ &= \| x_0^{-1} x - x_0^{-1} x_0 \| \\ &= \| x_0^{-1} (x - x_0) \| \\ &\leq \| x_0^{-1} \| \, \| x - x_0 \| \\ &< \| x_0^{-1} \| \frac{1}{\| x_0^{-1} \|} \\ &= 1 . \end{aligned}$$

Thus $\| z \| < 1$. So $e - z$ is invertible in A (Theorem 2.2.1) , and hence $e - z = y \in A^{-1}$.

Now , we have $x_0 , y \in A^{-1}$. So $x_0 y \in A^{-1}$ (Theorem 1.5) .

Therefore

$$\begin{aligned} x_0 y &= x_0 x_0^{-1} x \\ &= e x \\ &= x \in A^{-1}. \end{aligned}$$

Hence A^{-1} is open .

Corollary 2.2.5

Let A be a complex Banach algebra with unit e .Then the set of all non-invertible elements is closed .

Proof

Since A^{-1} is open (Theorem 2.2.4) , and the set of all non-invertible elements is complement of A^{-1} , so it is closed .

Theorem 2.2.6 [14]

Let A be a complex Banach algebra with unit e . Let $x \in A^{-1}$ and $y \in A$ such that

$$\| x - y \| \leq \frac{1}{\| x^{-1} \|} .$$

Then $y \in A^{-1}$ and $\| x^{-1} - y^{-1} \| \leq \dfrac{\| x^{-1} \|^2 \| x - y \|}{1 - \| x^{-1} \| \| x - y \|}$.

Proof

Let $x \in A^{-1}$ and $y \in A$. Then

$$\begin{aligned} \| e - x^{-1} y \| &= \| x x^{-1} - x^{-1} y \| \\ &= \| x^{-1} (x - y) \| \\ &\leq \| x^{-1} \| \| x - y \| \\ &\leq 1 . \end{aligned}$$

So $x^{-1} y$ is invertible (Theorem 2.2.3) and has an inverse in A say z . Then

$$x^{-1} y z = e \qquad (1).$$

Multiplying (1) on the left by x , we have

$$x x^{-1} y z = x e \text{ and so } y z = x .$$

We obtain

$$y z x^{-1} = x x^{-1} = e .$$

Hence $y z x^{-1} = e$.

Again multiplying (1) on the right by x^{-1} , we have

$(x^{-1} y z) x^{-1} = e x^{-1}$ and so $x^{-1} (y z x^{-1}) = x^{-1}$.

It follows that $z x^{-1} = \dfrac{1}{y}$, and we can obtain

$$z x^{-1} y = \frac{1}{y} y$$

$$= e .$$

Thus $z\,x^{-1}$ is the inverse of y and (Theorem 2.2.3), gives us

$$z = \sum_{n=0}^{\infty} (e - x^{-1} y)^n$$

$$= \sum_{n=0}^{\infty} (x^{-1}x - x^{-1} y)^n .$$

$$= \sum_{n=0}^{\infty} \left(x^{-1} (x - y) \right)^n .$$

We have

$$\| x^{-1} - y^{-1} \| = \| x^{-1} - z\, x^{-1} \|$$

$$= \| x^{-1} (e - z) \|$$

$$\leq \| e - z \| \, \| x^{-1} \|$$

$$\leq \| x^{-1} \| \sum_{n=1}^{\infty} \| x^{-1} \|^n \, \| x - y \|^n$$

$$\leq \| x^{-1} \| \sum_{n=1}^{\infty} \left(\| x^{-1} \| \, \| x - y \| \right)^n$$

$$= \frac{\| x^{-1} \|^2 \, \| x - y \|}{1 - \| x^{-1} \| \, \| x - y \|} .$$

Theorem 2.2.7

Let A be a complex Banach algebra with unit e. Let $x \in A^{-1}$ and $a \in A$ such that $\| a \| \leq \frac{1}{2} \| x^{-1} \|^{-1}$. Then $x + a \in A^{-1}$.

Proof

Let $x \in A^{-1}$, $a \in A$ and $\| a \| \leq \frac{1}{2} \| x^{-1} \|^{-1}$.

Then

$$\| x^{-1} a \| < \frac{1}{2} .$$

Hence $e + x^{-1} a \in A^{-1}$ (Theorem 2.2.2) , and so writing

$$x + a = x (e + x^{-1} a).$$

Now , we have $x \in A^{-1}$ and $e + x^{-1} a \in A^{-1}$. Thus $x (e + x^{-1} a) \in A^{-1}$.

Hence $x + a \in A^{-1}$.

Theorem 2.2.8

Let A be a complex Banach algebra with unit e . Let $x \in A^{-1}$ such that $\| x^{-1} \| = \frac{1}{\alpha}$, $h \in A$ and $\| h \| = \beta < \alpha$ .Then $x + h \in A^{-1}$ and

$$\| (x + h)^{-1} - x^{-1} + x^{-1} h x^{-1} \| \leq \frac{\beta^2}{\alpha^2 (\alpha - \beta)} .$$

Proof

Let $x \in A^{-1}$, $h \in A$. Then

$$\| x^{-1} h \| \leq \frac{\beta}{\alpha} < 1 .$$

Hence $e + x^{-1} h \in A^{-1}$ (Theorem 2.2.2) .

Since $x + h = x (e + x^{-1} h)$, so we have $x + h \in A^{-1}$.

Then

$$(x + h)^{-1} = (x (e + x^{-1} h))^{-1}$$

$$= (e + x^{-1} h)^{-1} x^{-1}.$$

Now , we have

$$(x + h)^{-1} - x^{-1} + x^{-1} h x^{-1} = [(e + x^{-1} h)^{-1} - e + x^{-1} h] x^{-1} .$$

Therefore

$$\| (x + h)^{-1} - x^{-1} + x^{-1} h x^{-1} \| = \| \left((e + x^{-1} h)^{-1} - e + x^{-1} h \right) x^{-1} \|$$

$$\leq \| (e + x^{-1} h)^{-1} - e + x^{-1} h \| \| x^{-1} \| .$$

It follows from (Theorem 2.2.2) with $x^{-1} h$ in place of x :

$$\left\| (x+h)^{-1} - x^{-1} + x^{-1} h x^{-1} \right\| \le \frac{\| x^{-1} h \|^2}{1 - \| x^{-1} h \|} \| x^{-1} \|$$

$$\le \frac{\frac{\beta^2}{\alpha^2} \; \frac{1}{\alpha}}{1 - \frac{\beta}{\alpha}}$$

$$= \frac{\beta^2}{\alpha^2 \; (\alpha - \beta)} \; .$$

Theorem 2.2.9

Let A be a complex Banach algebra with unit e. Let $x \in A$ and $\lambda \in \mathbb{C}$ such that $\| x \| < | \lambda |$. Then $x - \lambda e \in A^{-1}$.

Proof

Let $\| x \| < | \lambda |$. Then $\frac{\| x \|}{| \lambda |} < 1.$

So we obtain $\| \frac{x}{\lambda} \| < 1$.

Then $e - \lambda^{-1} x$ is invertible (Theorem 2.2.1) . Since

$-\lambda (e - \lambda^{-1} x) = x - \lambda e$, so $x - \lambda e$ is invertible .

Hence $x - \lambda e \in A^{-1}$.

Theorem 2.2.10

Let A be a commutative Banach algebra with unit . Let $a \in A$. Then the inversion mapping $a \to a^{-1}$ is continuous in A .

Proof

Suppose $x_n \in A^{-1}$ and $x_n \to a$ in A. We will show that $x_n^{-1} \to a^{-1}$ as $n \to \infty$. Let $a \in A$ such that

$$\| x_n - a \| \le \frac{1}{2 \| a^{-1} \|} .$$

Then

$$\| x_n^{-1} - a^{-1} \| = \| x_n^{-1} (a - x_n) a^{-1} \|$$

$$\le \| x_n^{-1} \| \, \| a - x_n \| \, \| a^{-1} \| \qquad (1)$$

$$\le \frac{1}{2} \| x_n^{-1} \| .$$

Since

$$\| x_n^{-1} \| - \| a^{-1} \| \le \| x_n^{-1} - a^{-1} \| ,$$

So

$$\| x_n^{-1} \| - \| a^{-1} \| \le \frac{1}{2} \| x_n^{-1} \| .$$

It follows that

$$\| x_n^{-1} \| \le 2 \| a^{-1} \| .$$

By (1) , we can get

$$\| x_n^{-1} - a^{-1} \| \le 2 \| a^{-1} \|^2 \| a - x_n \| \to 0 \quad (n \to \infty).$$

Thus $x_n^{-1} \to a^{-1}$.

Theorem 2.2.11

Let A be a commutative complex Banach algebra with unit . Let $a \in A$. Then the inversion mapping $a \to a^{-1}$ is a homeomorphism of A^{-1} to itself .

Proof

Clearly the mapping $a \to a^{-1}$ is onto . Let $a_1 , a_2 \in A$ with $a_1^{-1} = a_2^{-1}$.

Then

$$(a_1^{-1})^{-1} = (a_2^{-1})^{-1} ,$$

and so $a_1 = a_2$. Thus $a \to a^{-1}$ is one-one .

We have $a \to a^{-1}$ is continuous (Theorem 2.2.10) , and the inverse map from A onto A is continuous too .

Hence $a \to a^{-1}$ is homeomorphism .

Theorem 2.2.12

Let A be a commutative Banach algebra with unit e. Let (a_n) be a sequence in A^{-1} such that $a_n \to a$ in A as $n \to \infty$. If there exists a positive integer M such that $\| a_n^{-1} \| \le M$ for all $n \in \mathbb{N}$, then $a \in A^{-1}$ and

$$a_n^{-1} \to a^{-1} \quad as \quad n \to \infty.$$

Proof

Let $M > 0$ and let $a_n \to a$ as $n \to \infty$. Then (a_n) is a Cauchy sequence. Then for each $\varepsilon > 0$ there exists a positive integer N such that

$$\| a_n - a_m \| < \frac{\varepsilon}{M^2} \quad \text{for all} \quad n, m > \mathbb{N}.$$

Therefore

$$\begin{aligned}
\| a_n^{-1} - a_m^{-1} \| &= \| a_n^{-1} (a_n - a_m) a_m^{-1} \| \\
&\le \| a_n^{-1} \| \| a_n - a_m \| \| a_m^{-1} \| \\
&\le M^2 \frac{\varepsilon}{M^2} \\
&= \varepsilon .
\end{aligned}$$

Hence (a_n^{-1}) is Cauchy sequence in A. Since A is a Banach algebra, so a_n^{-1} converges to an element in A, say x. Then

$$x = \lim_{n \to \infty} (a_n^{-1}) .$$

So

$$\begin{aligned}
x\, a &= \lim_{n \to \infty} (a_n^{-1})(a_n) \\
&= e .
\end{aligned}$$

Hence a is invertible in A and $x = a^{-1}$.

Thus $a \in A^{-1}$ and $a_n^{-1} \to a^{-1}$ as $n \to \infty$.

Theorem 2.2.13

Let A be a complex Banach algebra with unit. Let x be a boundary point of A. Let $x_n \in A^{-1}$ such that $x_n \to x \ (n \to \infty)$ in A. Then

$$\| x_n^{-1} \| \to \infty \quad (n \to \infty).$$

Proof

If the conclusion is false, then there exists $M < \infty$ such that

$$\| x_n^{-1} \| < M \quad \text{for all } n.$$

Let x be a boundary point of A and let $x_n \to x \ (n \to \infty)$. Then for each $\varepsilon > 0$, there exists $N > 0$ such that

$$\| x_n - x \| < \varepsilon \quad (n > N).$$

Choose $\varepsilon = \dfrac{1}{M}$. Then

$$\| x_n - x \| < \frac{1}{M}.$$

$$\begin{aligned} \| e - x_n^{-1} x \| &= \| x_n^{-1} (x_n - x) \| \\ &\le \| x_n^{-1} \| \| x_n - x \| \\ &< M \cdot \frac{1}{M} \\ &= 1 \end{aligned}$$

Thus $\| e - x_n^{-1} x \| < 1$. So $x_n^{-1} x \in A^{-1}$. Then

$$x = x_n (x_n^{-1} x) \in A^{-1}.$$

We have $x \in A^{-1}$ and $x \in \partial (A)$.

It follows that $A^{-1} \cap \partial (A) \neq \emptyset$.

This is contradicts to A^{-1} is open (Theorem 2.2.4).

Hence $\| x_n^{-1} \| \to \infty \quad (n \to \infty)$.

Theorem 2.2.14

Let A *be a complex Banach algebra with unit* $e = 1$ *. Let* $(a_n) \subseteq A^{-1}$ *and* $a_n \to a \quad (n \to \infty)$ *in* A *. Then there exists a sequence* $(b_n) \subseteq A$ *with* $\| b_n \| = 1$ *and* $b_n a \to 0 \quad (n \to \infty)$ *.*

Proof

Set

$$b_n = \frac{a_n^{-1}}{\| a_n^{-1} \|} .$$

Then $\| b_n \| = 1$ and so (b_n) is a bounded sequence .

Also , $b_n a_n = \dfrac{1}{\| a_n^{-1} \|} \to 0$.

We have

$$b_n (a - a_n) \to 0 .$$

Adding , we obtain

$$b_n a \to 0 \qquad (n \to \infty) .$$

Definition 2.2.1

Let A be a complex Banach algebra with unit . We define the *exponential function* $\exp : A \to A$ by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad (x \in A) ,$$

and $\exp(0) = 1$.

Theorem 2.2.15

Let A *be a commutative Banach algebra with unit* $e = 1$ *. Let* $x , y \in A$ *. Then*

$$(\mathrm{i}) \quad \exp(x + y) = \exp(x) \exp(y) .$$

$$(\mathrm{ii}) \quad \exp(x) \in A^{-1} \ \textit{and}$$

$$(\exp(x))^{-1} = \exp(-x) .$$

Proof

Let $x, y \in A$. Then

(i) $\exp(x+y) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j!(n-j)!} x^{n-j} y^j$$

$$= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{j!\, n!} x^n y^j$$

$$= \exp(x) \exp(y).$$

(ii) Take $y = -x$ in (i). Then

$$\exp(0) = \exp(x) \exp(-x)$$

$$1 = \exp(x) \exp(-x).$$

Thus $(\exp(x))^{-1} = \exp(-x)$.

Theorem 2.2.16 [3]

Let A be a complex Banach algebra with unit $e = 1$. Let $x \in A$ such that $\| 1 - x \| < 1$. Then there exists $y \in A$ such that $\exp(y) = x$.

Definition 2.2.2

Let A be a complex Banach algebra with unit. We define

$$\exp(A) = \{ \exp(x) : x \in A \}.$$

It is clear that $\exp(A) \subset A^{-1}$.

Theorem 2.2.17

Let A be a commutative Banach algebra with unit $e = 1$. Then $\exp(A)$ is open in A^{-1}.

Proof

Let $x \in \exp(A)$. Then

$$x = \exp(h) \qquad (h \in A).$$

Let $y \in A$ with $\| x - y \| < \dfrac{1}{\| x^{-1} \|}$.

Then

$$\| 1 - x^{-1} y \| = \| x^{-1} \| \, \| x - y \|$$

$$\leq \| x^{-1} \| \frac{1}{\| x^{-1} \|}$$

$$= 1 .$$

By Theorem 2.2.16 , there exists $z \in A$ such that $x^{-1} y = \exp(z)$.

We have

$$y = \exp(h) \exp(z)$$

$$= \exp(h + z) \in \exp(A).$$

Hence $\exp(A)$ is open in A^{-1} .

2.3 Spectrum and Spectral radius of Banach algebras

Definition 2.3.1

Let A be a complex Banach algebra with unit e .The *spectrum* of an element $x \in A$, denoted by $\sigma_A(x)$, is defined by

$$\sigma_A(x) = \{ \lambda \in \mathbb{C} : x - \lambda e \notin A^{-1} \}.$$

The complement of $\sigma_A(x)$ in $\mathbb{C}$ is called the *resolvent set* of x . It is denoted by $\rho_A(x)$. That is

$$\rho_A(x) = \mathbb{C} \setminus \sigma_A(x).$$

Remark

Let A be a Banach algebra with unit . It is clear that x is invertible in A if and only if $0 \notin \sigma_A(x)$.

Example 2.3.1

Let $A = M_{2 \times 2}$ with complex entries .

Then $A = M_{2 \times 2}$ is a complex Banach algebra with unit $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in M_{2\times 2}$.

By an elementary theorem of matrix algebra it is known that $x - \lambda I$ has no inverse if and only if $\det(x - \lambda I) = 0$.

Then

$$\sigma_A(x) = \left\{ \lambda \in \mathbb{C} : \det \left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0 \right\}$$

$$= \left\{ \lambda \in \mathbb{C} : \det \left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = 0 \right\}$$

$$= \left\{ \lambda \in \mathbb{C} : \det \begin{pmatrix} -\lambda & i \\ -i & -\lambda \end{pmatrix} = 0 \right\}$$

$$= \left\{ \lambda \in \mathbb{C} : \lambda^2 + i^2 = 0 \right\}$$

$$= \{-1 , +1\} .$$

Lemma 2.3.1

Let A be a complex Banach algebra with unit e . Then

$$\sigma_A(0) = \{0\}.$$

Proof

$$\sigma_A(0) = \{ \lambda \in \mathbb{C} : 0 - \lambda e \notin A^{-1} \}$$

$$= \{ \lambda \in \mathbb{C} : -\lambda e \notin A^{-1} \}$$

$$= \{ \lambda \in \mathbb{C} : -\lambda \notin A^{-1} \}$$

$$= \{ 0 \} .$$

Theorem 2.3.2

Let A be a complex Banach algebra with unit e . Let $x \in A$. Then $\sigma_A(x)$ is non-empty .

Proof

Suppose for a contradiction that $x \in A$ has an empty spectrum .

Define

$$u(\lambda) = (x - \lambda e)^{-1} \quad (\lambda \in \mathbb{C}) .$$

Then u is well-defined and a continuous mapping of $\mathbb{C}$ into A .

Let $\lambda_0 \in \mathbb{C}$. Then

$$u(\lambda) - u(\lambda_0) = (x - \lambda e)^{-1} - (x - \lambda_0 e)^{-1}$$

$$= u(\lambda)\, u(\lambda_0)\, ((x - \lambda_0 e) - (x - \lambda e))$$

$$= (\lambda - \lambda_0)\, e\, u(\lambda)\, u(\lambda_0)$$

$$= (\lambda - \lambda_0)\, u(\lambda)\, u(\lambda_0) .$$

It follows that

$$\frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} = u(\lambda)\, u(\lambda_0) .$$

So

$$\lim_{\lambda \to \lambda_0} \frac{u(\lambda) - u(\lambda_0)}{\lambda - \lambda_0} = (u(\lambda_0))^2 \qquad (1)$$

Let f be a continuous linear functional on A . We define a function h by

$$h(\lambda) = f(u(\lambda)) \qquad (\lambda \in \mathbb{C}).$$

Since f and u are continuous , so is h .

Applying f to (1) , we thus obtain

$$\lim_{\lambda \to \lambda_0} \frac{h(\lambda) - h(\lambda_0)}{\lambda - \lambda_0} = f\left(u(\lambda_0)^2 \right).$$

Then h is an entire function from $\mathbb{C}$ into $\mathbb{C}$.

Since

$$u(\lambda) = -\lambda^{-1} (e - \lambda^{-1} x)^{-1} ,$$

and

$$(e - \lambda^{-1} x)^{-1} \to e^{-1} = e \quad \text{as} \quad |\lambda| \to \infty,$$

we obtain

$$|h(\lambda)| = \left| f(u(\lambda)) \right|$$

$$\leq \| f \| \, \| u(\lambda) \|$$

$$= \frac{1}{|\lambda|} \| f \| \, \| (e - \frac{1}{\lambda} x)^{-1} \|$$

$$\to 0 \quad \text{as} \quad |\lambda| \to \infty. \qquad (2)$$

This shows that h would be bounded on $\mathbb{C}$.

By Liouville's theorem , h is constant which is zero by (2). Then $h(\lambda) = f(u(\lambda)) = 0$. It follows that $u(\lambda) = 0$. So

$$\begin{aligned} \| e \| &= \| (x - \lambda e)(x - \lambda e)^{-1} \| \\ &= \| (x - \lambda e) \, u(\lambda) \| \\ &= \| 0 \| \\ &= 0, \end{aligned}$$

and contradicts to $\| e \| = 1$.

Hence $\sigma_A(x) \neq \emptyset$.

Remark

If A be a real Banach algebra with unit , then it is possible that there exists $x \in A$ such that $\sigma(x) = \emptyset$.

Example 2.3.2

Let $A = M_{2x2}$ be a real Banach algebra with unit $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $x = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_{2x2}$. Then

$$\sigma_A(x) = \left\{ \lambda \in \mathbb{R} : \det \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0 \right\}$$

$$= \left\{ \lambda \in \mathbb{R} : \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0 \right\}$$

$$= \left\{ \lambda \in \mathbb{R} : \lambda^2 + 1 = 0 \right\}$$

$$= \varnothing .$$

Lemma 2.3.3 [16]

Let A be a complex Banach algebra with unit . Let $x \in A$. The resolvent set $\rho_A(x)$ of x is open in $\mathbb{C}$.

Theorem 2.3.4

Let A be a Banach algebra with unit e . Let $x \in A$. Then $\sigma_A(x)$ is a compact subset of $\mathbb{C}$.

Proof

By the Heine-Borel Theorem (Theorem 1.26) it is enough to show that $\sigma_A(x)$ is bounded and closed . Let $\lambda \in \sigma_A(x)$. Then $x - \lambda e \notin A^{-1}$.

By Theorem 2.2.9 $\| x \| \geq | \lambda |$. So

$$\sigma_A(x) \subseteq \{ \lambda \in \mathbb{C} : | \lambda | \leq \| x \| \}.$$

Thus $\sigma_A(x)$ is bounded .

Since $\rho_A(x)$ is open in $\mathbb{C}$ (Lemma 2.3.3) , so $\sigma_A(x)$ is closed .

Theorem 2.3.5

Let A be a complex Banach algebra with unit $e = 1$. Let $x \in A$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. If $\lambda \in \sigma_A(x)$, then $\lambda^n \in \sigma_A(x^n)$.

Proof

Let $x \in A$ and let $\lambda \in \mathbb{C}$. Assume $\lambda^n \notin \sigma(x^n)$.

We have

$$(x^n - \lambda^n e) = (x - \lambda e)(x^{n-1} + \lambda x^{n-2} + \ldots + \lambda^{n-1} e) \rightarrow (1)$$

If multiply both sides of (1) by $(x^n - \lambda^n e)^{-1}$, then $(x - \lambda e)$ is invertible in A . So $\lambda \notin \sigma(x)$.

This completes the proof .

Theorem 2.3.6 [16]

Let A be a complex Banach algebra with unit e. Let B be a closed subalgebra of A containing e. If $x \in B$, then

$$\sigma_A(x) \subseteq \sigma_B(x),$$

and

$$\partial(\sigma_B(x)) \subseteq \partial(\sigma_A(x)).$$

Theorem 2.3.7 [12]

Let A be a closed subalgebra of a complex Banach algebra B. Let $x \in A$. If $\sigma_A(x)$ has empty interior, then

$$\sigma_A(x) = \sigma_B(x).$$

Theorem 2.3.8 [3]

Let A be a commutative complex Banach algebra with unit. Let $x \in A$. Then

$$\sigma_A(\exp(x)) = \exp(\sigma_A(x)).$$

Remark

In fact, there are some non-zero elements of complex Banach algebras which are not invertible. For examples :

(i) Let $A = M_{2\times 2}$ with complex entries. Then $M_{2\times 2}$ is a complex Banach algebra with unit $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in M_{2\times 2}$. Then x is a non-zero element of $M_{2\times 2}$ but x is not invertible.

(ii) Let $A = C[0,1]$.

Then $C[0,1]$ is a complex Banach algebra with unit $e = 1$.

Define f by

$$f(x) = \begin{cases} 0 & , \quad 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} & , \quad \frac{1}{2} \le x \le 1 \end{cases} .$$

Then f is a non-zero element of $C[0,1]$ but f is not invertible .

Proposition 2.3.9 [1]

Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible . Let $x \in A$. Then there exists a unique $\lambda \in \mathbb{C}$ such that $x = \lambda e$.

Proof

Let $x \in A$. Then $\sigma_A(x) \neq \varnothing$ (Theorem 2.3.2) . Hence there exists $\lambda \in \sigma_A(x)$ such that $x - \lambda e$ is not invertible . So $x - \lambda e = 0$. Thus $x = \lambda e$.

For uniqueness , let $x = \lambda e$, $x = \mu e$ ($\mu \in \mathbb{C}$, $\lambda \neq \mu$) . Let $\alpha = \lambda - \mu \neq 0$. Then $\alpha e = 0$, and so $e = 0$ which is a contradiction .

Corollary 2.3.10

Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible . Then A is commutative .

Proof

Let $x , y \in A$. Then there exists unique $\lambda , \mu \in \mathbb{C}$ ($\lambda \neq \mu$) such that

$$x = \lambda e \quad , \quad y = \mu e \quad \text{(Proposition 2.3.9)} .$$

Then

$$\begin{aligned} x\,y &= (\lambda e)(\mu e) \\ &= (\lambda \mu) e \\ &= (\mu \lambda) e \\ &= (y\,x). \end{aligned}$$

Hence A is commutative .

Theorem 2.3.11 (Gelfand - Mazur) [10]

Let A be a complex Banach algebra with unit e in which each non-zero element in A is invertible . Then A is isomorphic to $\mathbb{C}$.

Theorem 2.3.12 (Spectral mapping theorem) [10]

Let A be a complex Banach algebra with unit , and $x \in A$. Let P be a polynomial function with complex coefficients in A . Then

$$\sigma_A(P(x)) = P(\sigma_A(x)) .$$

Lemma 2.3.13

Let A be a commutative Banach algebra with unit. Let $x \in A$ and P be a polynomial function such that $p(x) = 0$. Then $P(\sigma_A(x)) = \{0\}$.

Proof

Let $x \in A$. By spectral mapping theorem ,

$$\begin{aligned} P(\sigma_A(x)) &= \sigma_A(P(x)) \\ &= \sigma_A(0) \\ &= \{0\} \qquad (\text{Lemma } 2.3.1) . \end{aligned}$$

Definition 2.3.2

Let A be a complex Banach algebra with unit e . Let $x \in A$ .The *spectral radius* of x , denoted by $r_A(x)$, is defined by

$$r_A(x) = \sup\left\{ |\lambda| : \lambda \in \sigma_A(x) \right\} .$$

Remarks

(i) $0 \le r_A(x) < \infty$ for all x .

(ii) If $r_A(x) = 0$, then $0 \in \sigma_A(x)$.

Example 2.3.3

Let $A = M_{2\times 2}$ with complex entries .

Let $x = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in M_{2\times 2}$.

Then $\sigma_A(x) = \{-1, +1\}$.

So $r_A(x) = \sup\{|-1|, |1|\}$

$= 1$.

Lemma 2.3.14

Let A be a complex Banach algebra with unit e. Let $x \in A$. Then

$$r_A(x) \leq \| x \| .$$

Proof

If $|\lambda| \geq \| x \|$, then $\| \lambda^{-1} x \| < 1$.

So $e - \lambda^{-1} x$ is invertible (Theorem 2.2.1) .

Since

$$-\lambda (e - \lambda^{-1} x) = x - \lambda e ,$$

so $x - \lambda e$ is invertible in A. Thus $\lambda \notin \sigma_A(x)$.

So $\lambda \in \sigma_A(x)$ implies $|\lambda| < \| x \|$.

Taking supremum over $\lambda \in \sigma_A(x)$, we obtain

$$\sup_{\lambda \in \sigma_A(x)} (|\lambda|) \leq \| x \| .$$

Hence $r_A(x) \leq \| x \|$.

Lemma 2.3.15

Let A be a complex Banach algebra with unit e. Let $x \in A$, $n \in \mathbb{N}$. Then

$$r_A(x^n) = r_A(x)^n .$$

Proof

Let $x \in A$. Then

$$r_A(x) = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}.$$

Therefore

$$r_A(x^n) = \sup \{ |\lambda| : \lambda \in \sigma_A(x^n) \}.$$

The spectral mapping theorem gives us :

$$\sigma_A(P(x)) = P(\sigma_A(x))$$

$$= \{ P(\lambda) : \lambda \in \sigma_A(x) \} .$$

Let $P(x) = x^n$. Then

$$\sigma_A(x^n) = \{ \lambda^n : \lambda \in \sigma_A(x) \}.$$

It follows that

$$r_A\ (x^n) = \sup\{\,|\lambda|^n : \lambda \in \sigma_A(x)\,\}$$

$$= r_A(x)^n\ .$$

Theorem 2.3.16 (Spectral Radius Formula) [10]

Let A *be a complex Banach algebra with unit* e *. Let* $x \in A$ *. Then*

$$r_A\ (x) = \lim_{n\to\infty} \|\ x^n\ \|^{\frac{1}{n}} \quad (\ n = 1, 2, 3, \ldots).$$

$$= \inf_{n\ge 1}\ (\ \|\ x^n\ \|^{\frac{1}{n}}\ )\ .$$

Chapter Three

Character mappings on Banach algebras

3.1 Character mappings

Definition 3.1.1

Let A be a complex Banach algebra with unit . A non-zero linear functional ϕ from A onto $\mathbb{C}$ is called *character if*

$$\phi(xy) = \phi(x)\,\phi(y) \quad (x, y \in A).$$

That is , ϕ is a multiplicative linear functional on A .

Remark

A character mapping ϕ on a complex Banach algebra A is a scalar homomorphism of A onto $\mathbb{C}$.

Remark

Let A be a complex Banach algebra with unit . Let ϕ be a character mapping from A onto $\mathbb{C}$. By linearity of ϕ , we have

$$\phi(\alpha x + \beta y) = \alpha\,\phi(x) + \beta\,\phi(y) \quad (x, y \in A, \alpha, \beta \in \mathbb{C}).$$

We can see that

(i) $\phi(0) = 0$.

(ii) $\phi(\alpha x) = \alpha\,\phi(x)$ (by putting $\beta = 0$) .

(iii) Put $\alpha = -1$ in (ii) . Then

$$\phi(-x) = -\phi(x).$$

Thus ϕ is an odd function .

(iv) Let $\alpha = 1$, $\beta = -1$. Then

$$\phi(x - y) = \phi(x) - \phi(y).$$

(v) Let $x_i \in A$ and $\lambda_i \in \mathbb{C}$. Then

$$\phi\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \phi(\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n)$$

$$= \phi(\lambda_1 x_1) + \phi(\lambda_2 x_2) + ... + \phi(\lambda_n x_n)$$

$$= \lambda_1 \phi(x_1) + \lambda_2 \phi(x_2) + \ldots + \lambda_n \phi(x_n)$$

$$= \sum_{i=1}^{n} \lambda_i \phi(x_i) .$$

We give some examples of character mappings on some Banach algebras .

Examples 3.1

(i) Define ϕ on $\mathbb{C}$ by

$$\phi(z) = z \quad (z \in \mathbb{C}).$$

Clearly ϕ is a linear map .

Let $z_1, z_2 \in \mathbb{C}$ .Then

$$\phi(z_1 z_2) = z_1 z_2$$
$$= \phi(z_1)\, \phi(z_2) .$$

Hence ϕ is a character mapping .

(ii) For each $x \in [0,1]$, define ϕ on $C[0,1]$ by

$$\phi(f) = f(x) \quad (f \in C[0,1]) .$$

Then ϕ is a character mapping .

(iii) For each $x \in [0,1]$, define ϕ on $C^1[0,1]$ by

$$\phi(f) = f(x) \quad (f \in C^1[0,1]) .$$

Then ϕ is a character mapping .

(iv) Let $A(\mathcal{D})$ be the disc algebra . Define ϕ on $A(\mathcal{D})$ by

$$\phi(f) = f(0) \quad (f \in A(\mathcal{D})) .$$

Then ϕ is a character mapping .

(v) Let $a \in \ell^1$ and let λ be a complex number .

Define ϕ on ℓ^1 by

$$\phi_\lambda(a) = \sum_{n=-\infty}^{\infty} a_n \lambda^n .$$

Then ϕ_λ is a character mapping .

(vi) Let A be a complex Banach algebra with unit and ϕ be a character mapping on A . Define $\tilde{\phi}$ on $A^{\#}$ by

$$\tilde{\phi}((x, \lambda)) = \phi(x) + \lambda \quad (x \in A, \lambda \in \mathbb{C}).$$

Let $(x, \lambda_1), (y, \lambda_2) \in A^{\#}$ and $\alpha, \beta \in \mathbb{C}$. Then

$$\begin{aligned}
\tilde{\phi}\left(\alpha(x, \lambda_1) + \beta(y, \lambda_2)\right) &= \tilde{\phi}\left((\alpha x, \alpha\lambda_1) + (\beta y, \beta\lambda_2)\right) \\
&= \tilde{\phi}(\alpha x + \beta y, \alpha\lambda_1 + \beta\lambda_2) \\
&= \phi(\alpha x + \beta y) + \alpha\lambda_1 + \beta\lambda_2 \\
&= \alpha\phi(x) + \beta\phi(y) + \alpha\lambda_1 + \beta\lambda_2 \\
&= \alpha(\phi(x) + \lambda_1) + \beta(\phi(y) + \lambda_2) \\
&= \alpha\tilde{\phi}(x, \lambda_1) + \beta\tilde{\phi}(y, \lambda_2).
\end{aligned}$$

Then $\tilde{\phi}$ is linear .

Also , we have

$$\begin{aligned}
\tilde{\phi}\left((x, \lambda_1)(y, \lambda_2)\right) &= \tilde{\phi}\left(xy + \lambda_1 y + \lambda_2 x, \lambda_1\lambda_2\right) \\
&= \phi(xy + \lambda_1 y + \lambda_2 x) + \lambda_1\lambda_2 \\
&= \phi(xy) + \phi(\lambda_1 y) + \phi(\lambda_2 x) + \lambda_1\lambda_2 \\
&= \phi(x)\phi(y) + \lambda_1\phi(y) + \lambda_2\phi(x) + \lambda_1\lambda_2 \\
&= \left(\phi(x) + \lambda_1\right)\left(\phi(y) + \lambda_2\right) \\
&= \tilde{\phi}\left((x, \lambda_1)\right)\tilde{\phi}\left((y, \lambda_2)\right).
\end{aligned}$$

Hence $\tilde{\phi}$ is a character mapping .

Remark

There are some different Banach algebras with the same character mappings. We give some results concerning character mappings .

Proposition 3.1.1

Let ϕ be a character mapping on a complex Banach algebra A with unit e. Then $\phi(e)=1$. In particular, if $e=1$, then $\phi(1)=1$.

Proof

For some $x \in A$, $\phi(x) \neq 0$, so we have

$$\phi(x)=\phi(x\,e)=\phi(x)\,\phi(e).$$

Hence $\phi(e)=1$.

Lemma 3.1.2

Let ϕ be a character mapping on a complex Banach algebra with unit e. Then $\phi(\lambda)=\lambda \quad (\lambda \in \mathbb{C})$.

Proof

Let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\phi(\lambda) &= \phi(\lambda . e)\\ &= \lambda\,\phi(e)\\ &= \lambda . 1 \quad (\text{ Proposition 3.1.1 })\\ &= \lambda .\end{aligned}$$

Lemma 3.1.3 [3]

Let A be a complex Banach algebra with unit. Let $x \in A$ and ϕ be a character mapping on A. Then

$$\phi\big(\phi(x)\big)=\phi(x).$$

Proposition 3.1.4

Let ϕ be a character mapping on a complex Banach algebra A with unit e. If x is an invertible element of A, then $\phi(x) \neq 0$.

Proof

On contrary, suppose $\phi(x)=0$.

Let x be invertible element of A.

Then there exists $y \in A$ such that

$$x\,y=y\,x=e.$$

Therefore

$$\phi(xy) = \phi(e)$$
$$\phi(x)\,\phi(y) = 1 \quad (\text{Proposition } 3.1.1)$$
$$0 = 1, \text{ which is impossible .}$$

Theorem 3.1.5 (Gleason , Kahane , Zelazko) [3]

If ϕ is a linear functional on a complex Banach algebra A with unit e such that $\phi(e) = 1$, and $\phi(x) \neq 0$ for every invertible $x \in A$, then

$$\phi(xy) = \phi(x)\,\phi(y) \quad (x, y \in A).$$

That is , ϕ is a character mapping .

The next theorem , give us the existence of character mappings on complex Banach algebras .

Theorem 3.1.6 [3]

Let A be a complex commutative Banach algebra with unit . Then there exists at least one character mapping on A .

Remark

Theorem 3.1.6 is not true in the case of a real commutative Banach algebra with unit .

Lemma 3.1.7

Let ϕ be a character mapping on a complex Banach algebra A with unit . Let $x \in A$, $n \in \mathbb{N}$. Then

$$\phi(x^n) = (\phi(x))^n .$$

Proof

We use mathematical induction .

Let $n = 1$. Then $\phi(x^1) = \phi(x)^1$ is true .

Now, suppose it is true for $n = k$

$$\phi(x^k) = (\phi(x))^k .$$

Now , we shall prove that it is true for $n = k + 1$.

We have

$$\phi(x^{k+1}) = \phi(x^k x)$$
$$= \phi(x^k)\,\phi(x)$$

$$= (\phi(x))^k \phi(x)$$

$$= (\phi(x))^{k+1}.$$

Thus $\phi(x^{k+1}) = (\phi(x))^{k+1}$.

Hence $\phi(x^n) = (\phi(x))^n$.

Corollary 3.1.8

Let ϕ be a character mapping on a complex Banach algebra with unit. Let $x \in A$, $n \in \mathbb{N}$. If $\phi(x) = x$, then $\phi(x^n) = x^n$.

Proof

Let $x \in A$. Then

$$\phi(x^n) = (\phi(x))^n \qquad (\text{Lemma } 3.1.7)$$

$$= x^n.$$

Theorem 3.1.9 [3]

Let A be a complex Banach algebra with unit e. Let ϕ be a linear functional on A. Then ϕ is a character mapping if and only if $\phi(e) = 1$, and $\phi(x^2) = \phi(x)^2 \quad (x \in A)$.

Theorem 3.1.10

Let ϕ be a character mapping on complex Banach algebra A with unit. Let x be an invertible element in A such that $x^2 = x$. Then $\phi(x) = 1$.

Proof

Let $x \in A$. Then

$$\phi(x) = \phi(x^2)$$

$$= \phi(x)^2 \qquad (\text{Lemma } 3.1.7).$$

So

$$\phi(x) - \phi(x)^2 = 0,$$

and we get

$$\phi(x)\big(1 - \phi(x)\big) = 0.$$

Since $\phi(x) \neq 0$ (Proposition 3.1.4), so $1-\phi(x)=0$.

Hence $\phi(x)=1$.

Theorem 3.1.11

Let A be a complex Banach algebra with unit e. Let ϕ be a character mapping on A. Let x, y be invertible elements in A. Then

$$\text{(i)} \quad \phi(x^{-1}) = (\phi(x))^{-1}.$$

$$\text{(ii)} \quad \phi((xy)^{-1}) = \phi(y)^{-1}\,\phi(x)^{-1}.$$

Proof

(i) Let x be an invertible element in A. Then there exists $x^{-1} \in A$ such that $x x^{-1} = x^{-1} x = e$.

So $\phi(x x^{-1}) = \phi(e) = 1$.

Hence $\phi(x)\,\phi(x^{-1}) = 1$.

Thus $\phi(x^{-1}) = (\phi(x))^{-1}$.

(ii) Let $x, y \in A$. Then

$$\begin{aligned}
\phi((xy)^{-1}) &= \phi(y^{-1}x^{-1}) \\
&= \phi(y^{-1})\,\phi(x^{-1}) \\
&= \phi(y)^{-1}\,\phi(x)^{-1} \quad (\text{By (i)}).
\end{aligned}$$

Theorem 3.1.12

Let h be a homomorphism mapping from a complex Banach algebra A with unit onto a complex Banach algebra B with unit. If ϕ is a character mapping on B, then $\phi \circ h$ is a character mapping on A.

Proof

The linearity of $\phi \circ h$ follows by the linearity of h and ϕ.

Let $x, y \in A$. Then

$$\begin{aligned}
(\phi \circ h)(xy) &= \phi(h(xy)) \\
&= \phi(h(x)\,h(y)) \\
&= \phi(h(x))\,\phi(h(y)) \\
&= (\phi \circ h)(x)\,(\phi \circ h)(y).
\end{aligned}$$

This completes the proof .

Lemma 3.1.13

Let ϕ_1 *and* ϕ_2 *be character mappings on a complex Banach algebra* A *with unit* $e = 1$. *Then* ϕ_1 *and* ϕ_2 *are linear independent .*

Proof

Let c_1 , c_2 be constants . Suppose

$$c_1 \phi_1 + c_2 \phi_2 = 0 \qquad (1) .$$

Then $c_1 \phi_1 = - c_2 \phi_2$ and so

$$c_1 \phi_1 (1) = - c_2 \phi_2 (1) .$$

Since $\phi_1 (1) = \phi_2 (1) = 1$, so $c_1 = - c_2$.

Equation (1) becomes

$$c_1 (\phi_1 - \phi_2) = 0 .$$

Since $\phi_1 - \phi_2 \neq 0$, so we obtain $c_1 = 0$ and hence $c_2 = 0$.

Hence ϕ_1 and ϕ_2 are linear independent .

Theorem 3.1.14

Let ϕ_1 *and* ϕ_2 *be character mappings on a complex Banach algebra* A *with unit* $e = 1$. *If there exists a non-zero constant* c *such that* $\phi_1 = c \phi_2$, *then* $c = 1$.

Proof

For the technique of the proof we have two methods :

Method (1) :

Let $\phi_1 = c \phi_2$. Then

$$\phi_1 (1) = c \phi_2 (1) \text{, and so } c = 1 .$$

Method (2) :

Let $x \in A$. Then

$$\phi_1 (x^2) = \phi_1 (x)^2 .$$

We obtain

$$
\begin{aligned}
c\,\phi_2(x^2) &= \phi_1(x^2) \\
&= (\phi_1(x))^2 \\
&= (c\,\phi_2(x))^2 \\
&= c^2\,\phi_2(x^2).
\end{aligned}
$$

Therefore $(c - c^2)\,\phi_2(x^2) = 0$.

Since $\phi_2(x^2) \neq 0$, so $c - c^2 = 0$, $c(1-c) = 0$, since c is not zero, so $c = 1$.

Theorem 3.1.15 [5]

Let ϕ be character mapping on a complex Banach algebra with unit e. Then ϕ is continuous and $\|\phi\| = 1$.

Lemma 3.1.16

Let ϕ be character mapping on a complex Banach algebra with unit . Then ϕ is $1-1$.

Proof

Let $x, y \in A$ and $\phi(x) = \phi(y)$. Then $\phi(x - y) = 0$ and so by Theorem 1.13 and Theorem 3.1.15 we obtain $x - y = 0$ and so $x = y$.

Theorem 3.1.17

Let ϕ be character mapping on a complex Banach algebra with unit . Let $x_n \to x$ and $y_n \to y$ in A. Then

(i) $\phi(x_n) \to \phi(x)$.

(ii) $\phi(x_n \pm y_n) \to \phi(x \pm y)$.

(iii) $\phi(\alpha x_n) \to \phi(\alpha x)$ $(\alpha \neq 0)$.

(iv) $\phi(x_n y_n) \to \phi(x y)$.

Proof

The proof follows by the continuity of ϕ (Theorem 3.1.15).

Lemma 3.1.18

Let ϕ be character mapping on a complex Banach algebra A with unit . If (x_n) is a Cauchy sequence in A, then $\phi(x_n)$ is Cauchy in $\mathbb{C}$.

Proof

Let (x_n) be a Cauchy sequence in A . Then for each $\varepsilon > 0$ there exists a positive integer N such that

$$\| x_n - x_m \| < \varepsilon \quad (n, m > N) .$$

We have

$$\begin{aligned} \| \phi(x_n) - \phi(x_m) \| &= \| \phi(x_n - x_m) \| \\ &\leq \| \phi \| \| x_n - x_m \| \\ &= \| x_n - x_m \| \qquad \text{(Theorem 3.1.15)} \\ &< \varepsilon . \end{aligned}$$

Hence $\| \phi(x_n) - \phi(x_m) \| < \varepsilon$.

Thus $(\phi(x_n))$ is a Cauchy sequence *in* $\mathbb{C}$.

Proposition 3.1.19

Let ϕ be character mapping on a complex Banach algebra with unit . Let $x \in A$. Then

$$\phi(\exp(x)) \neq 0 .$$

Proof

The proof follows by Proposition 3.1.4 and Definition 2.2.2 .

Theorem 3.1.20 [13]

Let ϕ be a linear functional on a commutative complex Banach algebra A with unit such that $\phi(\exp(x)) \neq 0$ for all $x \in A$. Then ϕ is a character mapping on A .

Proposition 3.1.21

Let ϕ be character mapping on a complex Banach algebra A with unit . Then there no exist $x , y \in A$ such that $x + x\,y = y$ and $\phi(x) = 1$.

Proof

On contrary, suppose there exist $x, y \in A$ such that $x + xy = y$ and $\phi(x) = 1$. We have

$$1 + \phi(y) = \phi(x) + \phi(x)\phi(y)$$

$$= \phi(x + xy)$$

$$= \phi(y),$$

which is impossible.

Proposition 3.1.22

Let ϕ be character mapping on a complex Banach algebra with unit. Let $x \in A$ such that $\phi(x) = 1$. Then

$$\phi(a + ax) = 2\phi(a) \quad (a \in A).$$

Proof

Let $a, x \in A$. Then

$$\phi(a + ax) = \phi(a) + \phi(ax)$$

$$= \phi(a) + \phi(a)\phi(x)$$

$$= 2\phi(a).$$

3.2 Kernals of Character mappings

Definition 3.2.1

Let ϕ be a character mapping on a complex Banach algebra A with unit e. The *kernal* of ϕ, denoted by $\ker(\phi)$, is defined by

$$\ker(\phi) = \{x \in A : \phi(x) = 0\}.$$

Remarks

(1) Note that $0 \in \ker(\phi)$ since $\phi(0) = 0$. So $\ker(\phi)$ is non-empty.

(2) $\ker(\phi)$ is a subspace of A.

Theorem 3.2.1

Let ϕ be character mapping on a complex Banach algebra A with unit .

(i) *If* $x \in \ker(\phi)$ *, then* $x^n \in \ker(\phi)$ $(n \in \mathbb{N})$.

(ii) *If* $a \in A$ *and* $x \in \ker(\phi)$ *, then* $\phi(ax) = 0$.

(iii) *If* $a \in A$ *,* $x \in A$ *such that* $\phi(x) = 1$ *, then* $a - ax \in \ker(\phi)$.

Proof

(i) Let $x \in \ker(\phi)$. Then $\phi(x) = 0$, since $\phi(x^n) = \phi(x)^n$, so $\phi(x^n) = 0$, and hence $x^n \in \ker(\phi)$.

(ii) Let $a \in A$, $x \in \ker(\phi)$. Then

$$\begin{aligned} \phi(ax) &= \phi(a)\,\phi(x) \\ &= 0 . \end{aligned}$$

(iii) Let $a \in A$. Then

$$\begin{aligned} \phi(a - ax) &= \phi(a) - \phi(ax) \\ &= \phi(a) - \phi(a)\,\phi(x) \\ &= 0 . \end{aligned}$$

Hence $a - ax \in \ker(\phi)$.

Lemma 3.2.2

Let ϕ be character mapping on a complex Banach algebra A with unit .

Let $a \in A$ *,* $x \in A \setminus \ker(\phi)$ *. Then* $a - \dfrac{\phi(a)}{\phi(x)}\, x \in \ker(\phi)$.

Proof

Let $a \in A$ and $x \in A \setminus \ker(\phi)$. Then

$$\begin{aligned} \phi\left(a - \frac{\phi(a)}{\phi(x)}\, x\right) &= \phi(a) - \phi\left(\frac{\phi(a)}{\phi(x)}\, x\right) \\ &= \phi(a) - \frac{\phi(a)}{\phi(x)}\,\phi(x) \\ &= 0 . \end{aligned}$$

It follows that

$$a - \frac{\phi(a)}{\phi(x)}\, x \in \ker(\phi).$$

Lemma 3.2.3

Let ϕ be a character mapping on a complex Banach algebra A with unit e. Let $x \in A$. Then

$$x - \phi(x)e \in \ker(\phi).$$

Proof

Let $x \in A$. Then

$$\phi(x - \phi(x)e) = \phi(x) - \phi(\phi(x)e)$$

$$= \phi(x) - \phi(x)\,\phi(e)$$

$$= \phi(x) - \phi(x)$$

$$= 0.$$

So $x - \phi(x)e \in \ker(\phi)$.

Notation

Let A be a complex Banach algebra with unit. Let Φ_A denote the set of all character mappings on A.

Theorem 3.2.4 [13]

Let A be a commutative complex Banach algebra with unit. Let M be a maximal ideal of A. Then there exists $\phi \in \Phi_A$ such that

$$M = \{x \in A : \phi(x) = 0\}.$$

Conversely, for any $\phi \in \Phi_A$, then

$$\{x \in A : \phi(x) = 0\} \text{ is a maximal ideal of } A.$$

Lemma 3.2.5

Let A be a complex Banach algebra with unit $e = 1$. Then $\lambda \in \sigma_A(x)$ if and only if $\phi(x) = \lambda$ for some $\phi \in \Phi_A$.

Proof

If $\lambda \notin \sigma_A(x)$, then there exists $y \in A$ such that

$$(x - \lambda e) y = 1 .$$

So it follows that

$$\phi\big((x - \lambda e) y\big) = \phi(1) ,$$

and so

$$\phi(x - \lambda e) \ \phi(y) = 1 .$$

Therefore

$$\phi(x - \lambda e) \neq 0 ,$$

$$\phi(x) - \lambda \ \phi(e) \neq 0 .$$

Hence $\phi(x) \neq \lambda$.

Remark

Let A be a complex Banach algebra with unit e . Let $x \in A$. Then

$$r_A(x) = \sup\{ |\lambda| : \lambda \in \sigma_A(x) \} .$$

Lemma 3.2.5 , gives us

$$r_A(x) = \sup_{\phi \in \phi_A} (|\phi(x)|) .$$

Lemma 3.2.6

Let A be a complex Banach algebra with unit e . Let $x \in A$ and $\phi \in \phi_A$ with $\phi(x) = 0$. Then $r_A(x) = 0$.

Proof

Let $x \in A$. Then

$$r_A(x) = \sup_{\phi \in \phi_A} (|\phi(x)|) .$$

Let $\phi(x) = 0$. Then

$$r_A(x) = 0 .$$

Theorem 3.2.7

Let A be a complex Banach algebra with unit e. Let $x, y \in A$ and $\lambda \in \mathbb{C}$. Then

$$(i) \quad r_A(\lambda x) = |\lambda| \, r_A(x).$$

$$(ii) \quad r_A(x+y) \le r_A(x) + r_A(y).$$

$$(iii) \quad r_A(xy) \le r_A(x) \, r_A(y).$$

Proof

Let $x, y \in A$ and $\lambda \in \mathbb{C}$. Then

(i) $$r_A(x) = \sup_{\phi \in \phi_A} (|\phi(x)|).$$

$$r_A(\lambda x) = \sup_{\phi \in \phi_A} (|\lambda \phi(x)|)$$

$$= \sup_{\phi \in \phi_A} (|\lambda| |\phi(x)|)$$

$$= |\lambda| \sup_{\phi \in \phi_A} (|\phi(x)|)$$

$$= |\lambda| \, r_A(x).$$

(ii) $$r_A(x+y) = \sup_{\phi \in \phi_A} (|\phi(x+y)|)$$

$$= \sup_{\phi \in \phi_A} (|\phi(x) + \phi(y)|)$$

$$\le \sup_{\phi \in \phi_A} (|\phi(x)|) + \sup_{\phi \in \phi_A} (|\phi(y)|)$$

$$= r_A(x) + r_A(y).$$

(iii) $$r_A(xy) = \sup_{\phi \in \phi_A} (|\phi(xy)|)$$

$$= \sup_{\phi \in \phi_A} (|\phi(x) \phi(y)|)$$

$$= \sup_{\phi \in \Phi_A} (|\phi(x)||\phi(y)|)$$

$$\leq \sup_{\phi \in \Phi_A} (|\phi(x)|) \sup_{\phi \in \Phi_A} (|\phi(y)|)$$

$$= r_A(x)\, r_A(y).$$

Definition 3.2.2

Let A be a commutative complex Banach algebra with unit . The *radical* of A is defined by

$$\text{rad}(A) = \bigcap_{\phi \in \Phi_A} \ker(\phi).$$

If $\text{rad}(A) = \{0\}$, then A is called *semi- simple* .

Examples 3.2 [3]

(i) $C^1[0,1]$ is a semi-simple Banach algebra .

(ii) The disc algebra $A(\mathcal{D})$ is a semi-simple Banach algebra .

(iii) ℓ^∞ = the space of all bounded sequences .

Then ℓ^∞ is a semi-simple Banach algebra .

Lemma 3.2.8

Let A be commutative complex Banach algebra with unit e. Let $x \in A$. Then x is in the radical of A if and only if $\phi(x) = 0$ for all $\phi \in \Phi_A$.

Proof

Let $x \in \text{rad}(A)$. Then

$$x \in \bigcap_{\phi \in \Phi_A} \ker(\phi),$$

if and only if $x \in \ker(\phi)$ for all $\phi \in \Phi_A$.

If and only if $\phi(x) = 0$.

Corollary 3.2.9

Let A be a commutative complex Banach algebra with unit e. Let $x \in A$. Then x is in the radical of A if and only if $r_A(x) = 0$.

Proof

Let $x \in \text{rad}(A)$ if and only if

$$\phi(x) = 0 \text{ for all } \phi \in \phi_A \quad (\text{Lemma 3.2.8}),$$

if and only if $r_A(x) = 0$.

Theorem 3.2.10 [3]

If $\psi : A \to B$ is homomorphism of a complex Banach algebra A with unit into a semi-simple commutative complex Banach algebra B with unit, then ψ is continuous.

3.3 The Gelfand transforms

Definition 3.3.1

Let A be a complex Banach algebra with unit. For each $x \in A$, we define the *Gelfand transform* x of x by

$$x(\phi) = \phi(x) \quad (\phi \in \phi_A).$$

Then x is a continuous complex - valued function from ϕ_A into $\mathbb{C}$.

We give some results concerning Gelfand transforms.

Lemma 3.3.1

Let A be a complex Banach algebra with unit. Then the Gelfand transform $x \to x$ is homomorphism.

Proof

Let $x, y \in A$, $\alpha \in \mathbb{C}$ and $\phi \in \phi_A$. Then

$$(\alpha x)\hat{}(\phi) = \phi(\alpha x)$$
$$= \alpha\,\phi(x)$$
$$= \alpha\, x(\phi).$$

and we have

$$(x+y)\hat{}\,(\phi) = \phi(x+y)$$

$$= \phi(x) + \phi(y)$$

$$= x(\phi) + y(\phi)$$

$$= (x+y)(\phi)\ .$$

Thus x is linear .

Also , $(x\ y)\hat{}\,(\phi) = \phi(x\ y)$

$$= \phi(x)\,\phi(y)$$

$$= x(\phi)\ y(\phi)$$

$$= (x\ y)(\phi).$$

Hence $x \rightarrow x$ is homomorphism .

Lemma 3.3.2

Let A be a complex Banach algebra with unit . Let $x \in A$. Then the Gelfand transform $x \rightarrow x$ is one-one .

Proof

Let $x(\phi_1) = x(\phi_2)$ $(\phi_1, \phi_2 \in \varphi_A)$. Then

$\phi_1(x) = \phi_2(x)$, and so $\phi_1 = \phi_2$.

Lemma 3.3.3

Let A be a complex Banach algebra with unit . If x is invertible in A , then $x(\phi) \neq 0$ for all $\phi \in \varphi_A$.

Proof

Let x be an invertible element in A . Then

$\phi(x) \neq 0$ for all $\phi \in \varphi_A$ (Proposition 3.1.4) .

Hence $x(\phi) \neq 0$.

Lemma 3.3.4

Let A be a complex Banach algebra with unit. Let $x \in A$ and $\phi \in \varphi_A$.

Then $x(\varphi_A) = \sigma_A(x)$.

Proof

Let $\phi \in \varphi_A$. Then

$$x(\varphi_A) = \{ x(\phi) : \phi \in \varphi_A \}$$

$$= \{ \phi(x) : \phi \in \varphi_A \}$$

$$= \sigma_A(x).$$

Lemma 3.3.5 [6]

Let A be a complex Banach algebra with unit. Let $x \in A$ and $\phi \in \varphi_A$.

Then

$$\| x \|_{\varphi_A} \leq \| x \|_A .$$

Proof

Let $x \in A$, $\phi \in \varphi_A$. Then

$$| x(\phi) | = | \phi(x) | \leq \| x \|$$

It follows that

$$\| x \|_{\varphi_A} \leq \| x \|_A .$$

Theorem 3.3.6

Let A be a commutative complex Banach algebra with unit. Let $x \in A$.

Then

$$r_A(x) = 0 \ \ \textit{if and only if} \ \ x = 0.$$

Proof

Let $r_A(x) = 0$. Then $\phi(x) = 0$.

$$x(\phi) = \phi(x) = 0.$$

Conversely, let $x(\phi) = 0$. Then

$$\phi(x) = 0 \text{ and so } r_A(x) = 0.$$

Chapter Four

Banach algebras with involutions

4.1 Banach star algebras

Definition 4.1.1

Let A be a complex algebra . A mapping $x \to x^*$ of A into A is called an *involution* on A if it has the following properties for all $x , y \in A , \lambda \in \mathbb{C}$:

(i) $(x + y)^* = x^* + y^*$

(ii) $(\lambda x)^* = \overline{\lambda} x^*$

(iii) $(x y)^* = y^* x^*$

(iv) $x^{**} = x$.

Axioms (i) and (ii) define a mapping $x \to x^*$ as linear conjugate .
Axiom (iv) implies that the involution is onto mapping .

Remarks

Let A be a complex algebra with involution $*$. Let $x , y \in A$. Then

(i) $x^{**} = (x^*)^*$.

(ii) $(x - y)^* = x^* - y^*$.

(iii) Let $i \in \mathbb{C}$ .Then $(x + i y)^* = x^* - i y^*$.

(iv) In general , $x x^* \neq x^* x$.

Definition 4.1.2

A complex algebra A with an involution $*$ is called a *star algebra* or an *algebra with involution* .

Remark

Let A be a star algebra . Then

$$0^{**} = 0.$$

Lemma 4.1.1

Let A be a star algebra . Then $0^{*} = 0$.

Proof

$$0^{*} = (\, 0^{*} \,.\, 0 \,)^{*}$$

$$= 0^{*} \,.\, 0^{**}$$

$$= 0^{*} . \, 0$$

$$= 0 \,.$$

Remark

Let A be a star algebra with unit e . Then $e^{**} = e$.

Lemma 4.1.2

Let A be a star algebra with unit e . Then $e^{*} = e$.

Proof

Let e be the identity element of A . Then

$$e^{*} = e \,.\, e^{*}$$

$$= e^{**} . \, e^{*}$$

$$= (\, e \,.\, e^{*} \,)^{*}$$

$$= (\, e^{*})^{*}$$

$$= e^{**}$$

$$= e \,.$$

Hence $e^{*} = e$.

Remark

In particular , if $e = 1$, then $1^{*} = 1$.

Examples 4.1

(1) Let $f \in C^1[0,1]$.

Define f^* on $C^1[0,1]$ by

$$f^* = \overline{f}.$$

Let $f, g \in C^1[0,1], \lambda \in \mathbb{C}$. Then

(i) $(f+g)^* = \overline{f+g} = \overline{f} + \overline{g} = f^* + g^*$.

(ii) $(\lambda f)^* = \overline{(\lambda f)} = \overline{\lambda}\ \overline{f} = \overline{\lambda}\ f^*$.

(iii) $(f\ g)^* = \overline{(f\ g)} = \overline{(g\ f)} = \overline{g}\ \overline{f} = g^* f^*$.

(iv) $f^{**} = (f^*)^* = (\overline{f})^* = \overline{\overline{f}} = f$.

Hence $f \to \overline{f}$ defines an involution on $C^1[0,1]$.

Thus $C^1[0,1]$ is a star algebra.

(2) We define an involution on $A(D)$ by

$$f^*(z) = \overline{f(\overline{z})} \quad (f \in A(D), z \in \mathbb{C}).$$

In the same way, $A(D)$ becomes a star algebra.

(3) Let $T \in BL(H)$ and $T^* \in BL(H)$ Hilbert space adjoint operator of T.

Let $T, S \in BL(H)$. Then

(i) $(T+S)^* = T^* + S^*$.

(ii) $(\lambda T)^* = \overline{\lambda}\ T^* \quad (\lambda \in \mathbb{C})$.

(iii) $(T\ S)^* = S^*\ T^*$

(iv) $(T^*)^* = T$

Hence $T \to T^*$ is an involution.

Thus $BL(H)$ is a star algebra.

(4) Let $A \in M_{n \times n}$.

Define A^* on $M_{n \times n}$ by

$$A^* = \overline{A^t}.$$

(The complex conjugate of transpose of A).

Let $A, B \in M_{n\times n}$. Then

(i) $(A+B)^* = \overline{(A+B)^t} = \overline{A^t + B^t} = \overline{A^t} + \overline{B^t} = A^* + B^*$.

(ii) $(\lambda A)^* = \overline{(\lambda A)^t} = \overline{\lambda A^t} = \overline{\lambda}\ \overline{A^t} = \overline{\lambda} A^*$ $\quad (\lambda \in \mathbb{C})$.

(iii) $(AB)^* = \overline{(AB)^t} = \overline{B^t A^t} = \overline{B^t}\ \overline{A^t} = B^* A^*$.

(iv) $A^{**} = (A^*)^* = (\overline{A^t})^* = A$.

Hence $A \to A^*$ is an involution.

Thus $M_{n\times n}$ is a star algebra.

(5) Let A be a commutative star algebra with unit e and an involution $*$.

Let $x \in A$, $\lambda \in \mathbb{C}$, and $(x, \lambda) \in A^{\#}$. Define

$$(x, \lambda)^* = x^* + \overline{\lambda}\, e.$$

Let $x_1, x_2 \in A$, $\lambda_1, \lambda_2 \in \mathbb{C}$ and $(x_1, \lambda_1), (x_2, \lambda_2) \in A^{\#}$. Then

(i) $((x_1, \lambda_1) + (x_2, \lambda_2))^* = ((x_1 + x_2, \lambda_1 + \lambda_2))^*$

$$= (x_1 + x_2)^* + \overline{(\lambda_1 + \lambda_2)}\, e$$

$$= x_1^* + x_2^* + (\overline{\lambda_1} + \overline{\lambda_2})\, e$$

$$= x_1^* + x_2^* + \overline{\lambda_1}\, e + \overline{\lambda_2}\, e$$

$$= (x_1^* + \overline{\lambda_1}\, e) + (x_2^* + \overline{\lambda_2}\, e)$$

$$= (x_1, \lambda_1)^* + (x_2, \lambda_2)^*.$$

(ii) Let $\lambda \in \mathbb{C}$. Then

$$(\lambda (x_1, \lambda_1))^* = (\lambda x_1, \lambda \lambda_1)^*$$

$$= (\lambda x_1)^* + (\overline{\lambda \lambda_1})\, e$$

$$= \overline{\lambda}\, x_1^* + (\overline{\lambda}\ \overline{\lambda_1})\, e$$

$$= \overline{\lambda}\, (x_1^* + \overline{\lambda_1}\, e)$$

$$= \overline{\lambda}\, (x_1, \lambda_1)^*.$$

(iii) $((x_1, \lambda_1)\ (x_2, \lambda_2))^* = (x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1, \lambda_1 \lambda_2)^*$

$$= (x_1 x_2 + \lambda_1 x_2 + \lambda_2 x_1)^* + (\overline{\lambda_1 \lambda_2})e$$

$$= (x_1 x_2)^* + (\lambda_1 x_2)^* + (\lambda_2 x_1)^* + (\overline{\lambda_1} \ \overline{\lambda_2})\ e$$

$$= x_2^* x_1^* + \overline{\lambda_1} x_2^* + \overline{\lambda_2} x_1^* + (\overline{\lambda_1} \ \overline{\lambda_2}\ e)$$

$$= (x_2^* + \overline{\lambda_2} e)(x_1^* + \overline{\lambda_1} e)$$

$$= (x_2, \lambda_2)^* (x_1, \lambda_1)^* .$$

(iv) We have

$$(x, \lambda)^{**} = ((x, \lambda)^*)^*$$

$$= (x^* + \overline{\lambda}\ e)^*$$

$$= x^{**} + (\overline{\lambda}\ e)^*$$

$$= x + \overline{\overline{\lambda}}\ e^*$$

$$= x + \lambda\ e$$

$$= (x, 0) + \lambda\ (0, 1)$$

$$= (x, \lambda) .$$

Hence $A^\#$ is a commutative star algebra with the given involution .

We shall state and prove some results concerning star algebras .

Lemma 4.1.3

Let A be a star algebra and $x \in A$. Then $x \to x^*$ is one – one .

Proof

Let $x_1, x_2 \in A$ and let $x_1^* = x_2^*$. Then

$$x_1^* - x_2^* = 0 .$$

Therefore

$$(x_1 - x_2)^* = 0^* \quad \text{(Lemma 4.1.1) .}$$

It follows that

$$(x_1 - x_2)^{**} = (0)^{**} .$$

So

$$x_1 - x_2 = 0.$$

Thus $x_1 = x_2$.

Theorem 4.1.4

Let A be a star algebra with unit e . Let $x \in A$. Then x is invertible if and only if x^* is invertible and $(x^*)^{-1} = (x^{-1})^*$.

Proof

Let x be invertible element in A . Then

$$x^{-1} x = x\, x^{-1} = e .$$

So

$$(x^{-1} x)^* = e^* = e \qquad \text{(Lemma 4.1.2)} .$$

Therefore

$$x^* (x^{-1})^* = e .$$

It follows that

$$(x^*)^{-1} = (x^{-1})^* .$$

Conversely , let x^* be an invertible element in A . Then

$$x^* (x^{-1})^* = e .$$

Thus $(x^{-1} x)^* = e$ and $(x^{-1} x)^{**} = e^*$.

It follows that $x^{-1} x = e$.

Hence x is invertible in A .

Lemma 4.1.5

Let A be a star algebra with unit e . If x is invertible in A , then $x x^*$ is invertible .

Proof

Let x be an invertible element in A . Then x^* is invertible (Theorem 4.1.4).

Hence $x\, x^*$ is invertible (Theorem 1.5) .

Remark

In the same way , we can prove that $x^* x$ is invertible .

Lemma 4.1.6

Let A be a star algebra with unit e. Let x be invertible in A. Then

$$x^* (x x^*)^{-1} = x^{-1}.$$

Proof

Let $x \in A$. Then

$$x^* (x x^*)^{-1} = x^* \left((x^*)^{-1} x^{-1} \right)$$

$$= (x^* (x^*)^{-1}) x^{-1}$$

$$= e^* x^{-1}$$

$$= e x^{-1}$$

$$= x^{-1}.$$

Lemma 4.1.7

Let A be a commutative star algebra and $x, y \in A$. Then

$$x^* y^* = y^* x^*.$$

Proof

$$x^* y^* = (y x)^*,$$

since A is commutative, so

$$x^* y^* = (x y)^*$$

$$= y^* x^*.$$

Remark

Let A be a commutative star algebra. Let $x, y \in A$. Then

$$(x y)^* = y^* x^* = x^* y^*.$$

Lemma 4.1.8

Let $n \in \mathbb{N}$. Let A be a star algebra and $x \in A$. Then

$$(x^n)^* = (x^*)^n.$$

Proof

We shall use mathematical induction

Let $n = 1$. Then

$$(x^1)^* = (x^*)^1 \qquad (x \in A).$$

Now, suppose it is true for $n = k$

$$(x^k)^* = (x^*)^k$$

We shall prove it is true for $n = k + 1$. We have

$$(x^{k+1})^* = (x^k x)^*$$

$$= x^* (x^k)^*$$

$$= x^* (x^*)^k$$

$$= (x^*)^{k+1}$$

Thus $(x^n)^* = (x^*)^n$.

Definition 4.1.3

A complex normed algebra A with an involution $*$ is called a *normed star algebra*.

Definition 4.1.4

A complete normed star algebra is called a *Banach star algebra*.

Remark

An involution on a Banach star algebra may or may not be continuous.

Theorem 4.1.9 [3]

Let A be a commutative Banach star algebra and semisimple. Then every involution is continuous.

Proposition 4.1.10 [5]

Let A be a Banach star algebra with unit. Then

$$\exp(x^*) = (\exp(x))^* \qquad (x \in A).$$

Proof

Let $x \in A$. Then

$$(\exp(x))^* = \sum_{n=0}^{\infty} \frac{(x^n)^*}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(x^*)^n}{n!} \qquad \text{(Lemma 4.1.8) .}$$

$$= \exp(x^*) .$$

Corollary 4.1.11

Let A be a Banach star algebra with unit . Let $a \in A$ and $\exp(x) = 1$.
Then $\exp(x^*) = 1$.

Proof

$$\exp(x^*) = (\exp(x))^* \qquad \text{(Proposition 4.1.10) .}$$

$$= (1)^*$$

$$= 1 .$$

Theorem 4.1.12

Let A be a Banach star algebra with unit e . Let $\lambda \in \mathbb{C}$, $x \in A$. Then $\lambda \in \sigma_A(x)$ if and only if $\overline{\lambda} \in \sigma_A(x^*)$.

Proof

Since x is invertible in A if and only if x^* is invertible (Theorem 4.1.4) , and

$$(x^*)^{-1} = (x^{-1})^* .$$

Let $\lambda \in \sigma_A(x)$. Then

$x - \lambda e$ is not invertible in A if and only if $(x - \lambda e)^*$ is not invertible in A. So

$$(x - \lambda e)^* = x^* - \overline{\lambda}\, e^*$$

$$= x^* - \overline{\lambda}\, e \quad \text{is not invertible in } A .$$

Hence $\overline{\lambda} \in \sigma_A(x^*)$.

4.2 Hermitian and Normal elements

Definition 4.2.1

Let A be a star algebra . An element $x \in A$ is called *hermitian* (or *self – adjoint*) if

$$x = x^* .$$

Examples 4.2.1

(i) 0 is hermitian since $0^* = 0$ (Lemma 4.1.1) .

(ii) e is hermitian since $e^* = e$ (Lemma 4.1.2) .

(iii) The identity operator I of $BL(H)$ is hermitian since $I^* = I$ (Theorem 1.32 (v)) .

Remark

Let A be a star algebra . Let $a_1 , a_2 , .. , a_n$ be hermitian elements in A . Then

$$a_1 = a_1^* , a_2 = a_2^* , ... , a_n = a_n^* \ .$$

Therefore

$$\sum_{n=1}^{k} a_n = \sum_{n=1}^{k} a_n^*$$

Lemma 4.2.1

Let $T \in BL(H)$. Then $(T^* T - I)$ is hermitian .

Proof

$$\begin{aligned} (T^* T - I)^* &= (T^* T)^* - I^* \\ &= T^* T^{**} - I^* \\ &= T^* T - I \ . \end{aligned}$$

Hence $(T^* T - I)$ is hermitian .

Lemma 4.2.2

Let A be a star algebra and $x \in A$. Then x is hermitian if and only if x^* is hermitian .

Proof

Let x be hermitian . Then

$$x^* = x .$$

So

$$(x^*)^* = x^{**} = x = x^* .$$

Hence x^* is hermitian .

Conversely , let x^* be hermitian .

Then

$$x^* = (x^*)^* .$$

So

$$x = x^{**} = (x^*)^* = x^* .$$

Hence x is hermitian .

Theorem 4.2.3

Let A be a star algebra and let $x , y \in A$ be hermitian . Let $\alpha , \beta \in \mathbb{R}$.

Then

(i) $x + y$

(ii) αx

(iii) $\alpha x + \beta y$

are hermitian .

Proof

(i) Let $x , y \in A$ be hermitian . Then

$$x^* = x \quad , \quad y^* = y .$$

Then

$$(x + y)^* = x^* + y^*$$
$$= x + y .$$

Hence $x + y$ is hermitian .

(ii) Let $x \in A$ and $\alpha \in \mathbb{R}$. Then

$$(\alpha x)^* = \overline{\alpha} x^*$$
$$= \alpha x .$$

Hence αx is hermitian .

(iii) The proof follows by (i) and (ii) .

Theorem 4.2.4

Let A be a commutative star algebra . Let x , y be hermitian elements in A . Then $x y$ is hermitian .

Proof

Let x , y be hermitian in A .

Then

$$x^* = x \quad \text{and} \quad y^* = y .$$

We have

$$\begin{aligned}(x y)^* &= y^* x^* \\ &= y x \\ &= x y .\end{aligned}$$

Lemma 4.2.5

Let A be a star algebra and $x , y \in A$. If $x y$ is hermitian and a non-zero element x is hermitian , then y is hermitian .

Proof

Let $x y$ be hermitian and x be a non-zero hermitian element in A .

Then

$$(x y)^* = x y$$

We have

$$\begin{aligned}(x y)^* &= y^* x^* \\ &= y^* x .\end{aligned}$$

We obtain

$$y^* x = x y .$$

It follows that $y^* = y$.

Hence y is hermitian .

Lemma 4.2.6

Let $n \in \mathbb{N}$ *and let* A *be a star algebra and* $x \in A$. *Let* x *be hermitian element. Then* x^n *is hermitian.*

Proof

The proof follows By mathematical induction.

Theorem 4.2.7

Let A *be a star algebra and* $x \in A$. *Then* $x + x^*$ *is hermitian.*

Proof

Let $x \in A$. Then

$$(x + x^*)^* = x^* + (x^*)^*$$
$$= x^* + x$$
$$= x + x^*.$$

Remark

Let $x \in A$. Then

$$(x - x^*)^* = x^* - x^{**}$$
$$= x^* - x.$$

Hence $x - x^*$ is not hermitian.

Theorem 4.2.8

Let A *be a star algebra and* $x \in A$. *Then* $x\,x^*$ *and* x^*x *are hermitian.*

Proof

$$(x\;x^*)^* = x^{**}\;x^*$$
$$= x\;x^*,$$

and also, we have

$$(x^*x)^* = x^*x^{**}$$
$$= x^*x.$$

Remarks

Let A be a star algebra and $x \in A$. Then

(i) ix is not hermitian since

$$(ix)^* = -ix^*.$$

(ii) $i(x - x^*)$ is hermitian since

$$(i(x - x^*))^* = (-i)(x - x^*)^*$$

$$= (-i)(x^* - x)$$

$$= i(x - x^*).$$

Remarks

Let A be a star algebra and $x \in A$. Let $u = \dfrac{x + x^*}{2}$.

Then

$$u^* = \frac{(x + x^*)^*}{2}$$

$$= \frac{x^* + x^{**}}{2}$$

$$= \frac{x + x^*}{2}$$

$$= u.$$

Hence u is a hermitian element in A.

Let $v = \dfrac{x - x^*}{2}$. Then

$$v^* = \frac{(x - x^*)^*}{2}$$

$$= \frac{x^* - x^{**}}{2}$$

$$= \frac{-(x - x^*)}{2}$$

$$= -v.$$

Hence v is not hermitian .

Let $i \in \mathbb{C}$. We have

$$\left(\frac{-i\,(x - x^*)}{2} \right)^* = \frac{i\,(x^* - x)}{2}$$

$$= \frac{-i\,(x - x^*)}{2} .$$

Hence $\frac{-i\,(x - x^*)}{2}$ is hermitian .

Theorem 4.2.9 [3]

Let A be a star algebra and $x \in A$. Then x has a unique representation

$$x = u + i\,v \qquad (u , v \in A),$$

where u and v are hermitian .

Proof

Let $u = \frac{x + x^*}{2}$ and $v = \frac{-i\,(x - x^*)}{2}$.

Then u and v are hermitian and we obtain

$$x = u + i\,v \qquad (u , v \in A) .$$

For uniqueness , suppose $x = u' + i\,v'$ (u' and v' are hermitian ,

($u' , v' \in A$) .Then

$$u + i\,v = u' + i\,v'$$

$$u - u' = i\,(v' - v) .$$

Put $w = v' - v$. Then $i\,w = u - u'$. By Theorem 4.2.3 .We get w and $i\,w$ are hermitian . We have

$$i\,w = (i\,w)^* = -i\,w^* = -i\,w .$$

Hence $w = 0$ and so $v = v'$ and $u = u'$.

This completes the proof .

Definition 4.2.2

Let A be a star algebra . An element $x \in A$ is called *normal* if

$$x\ x^* = x^*\ x\ .$$

Examples 4.2.2

(i) 0 is normal since

$$0\ 0^* = 0,$$

and

$$0^*\ 0 = 0\ .$$

(ii) The unit element e in a star algebra A is normal since

$$e\ e^* = e\ e = e\ ,$$

and

$$e^*\ e = e\ e = e\ .$$

(iii) Let $A \in M_{2\times 2}$.

Define A^* on $M_{2\times 2}$ by

$$A^* = \overline{A^t}\ .$$

Let $A = \begin{pmatrix} 1 & 1 \\ i & 3+2\,i \end{pmatrix} \in M_{2\times 2}$.

Then $A^* = \begin{pmatrix} 1 & -\,i \\ 1 & 3-2\,i \end{pmatrix}$.

So

$$A\ A^* = \begin{pmatrix} 1 & 1 \\ i & 3+2\,i \end{pmatrix} \begin{pmatrix} 1 & -\,i \\ 1 & 3-2\,i \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3-3\,i \\ 3+3\,i & 14 \end{pmatrix},$$

and

$$A^*\ A = \begin{pmatrix} 1 & -\,i \\ 1 & 3-2\,i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & 3+2\,i \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3-3i \\ 3+3i & 14 \end{pmatrix} .$$

Thus $A A^* = A^* A$.

Hence A is normal .

Let $B = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}$.

Then $B^* = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$.

So

$$B B^* = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix},$$

and

$$B^* B = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Since $B B^* \neq B^* B$, so B is not normal .

Theorem 4.2.10

Let A be a star algebra with unit and $x \in A$. Then x is normal if and only if x^{-1} is normal .

Proof

Let x be normal in A. Then

$$(x^{-1})^* \; x^{-1} = (x^*)^{-1} \; x^{-1} \qquad \text{(Theorem 4.1.4)}$$

$$= (x \; x^*)^{-1}$$

$$= (x^* x)^{-1}$$

$$= x^{-1} (x^*)^{-1}$$

$$= x^{-1} (x^{-1})^* .$$

Hence x^{-1} is normal .

Conversely , let x^{-1} be normal . Then $(x^{-1})^{-1}$ is normal .

Hence x is normal .

Lemma 4.2.11

Let A be a star algebra and $x \in A$. If x is hermitian , then x is normal .

Proof

Let x be hermitian in A . Then

$$x x^* = x x$$
$$= x^2$$

and

$$x^* x = x x$$
$$= x^2 .$$

Remark

Note that , normal element in a star algebra A need not be hermitian . For example :

Define $T \in BL(H)$ by

$$T = 2 i I ,$$

where $I : H \to H$ is the identity operator . Then

$$T^* = -2 i I ,$$

and so

$$T T^* = T^* T = 4 I .$$

Hence T is normal .

But $T \neq T^*$. So T is not hermitian .

4.3 B*-algebras

Definition 4.3.1

Let A be a Banach star algebra such that

$$\| x^* x \| = \| x \|^2 \quad (x \in A).$$

Then A is called *a* B^**– algebra* .

Examples 4.3

(i) Let X be a compact Hausdorff space . Let $C(X)$ denote the algebra of all complex – valued continuous functions on X .

The norm on $C(X)$ is given by

$$\| f \| = \sup_{x \in X} (| f(x) |) \quad (f \in C(X)) .$$

The involution on $C(X)$ is given by

$$f^* = \overline{f} .$$

Let $f \in C(X)$. Then

$$\begin{aligned} \| f^* f \| &= \sup_{x \in X} (| \overline{f}(x) f(x) |) \\ &= \sup_{x \in X} (| f(x) |^2) \\ &= \Big(\sup_{x \in X} (| f(x) |) \Big)^2 \\ &= \| f \|^2 . \end{aligned}$$

Thus $C(X)$ is a B^*– algebra .

(ii) Let H be a complex Hilbert space . Let $T \in BL(H)$ and Let T^* be the Hilbert space adjoint of T .

Then $T \to T^*$ is an involution on $BL(H)$. Then

$$\| T^* T \| = \| T \|^2 .$$

Hence $BL(H)$ is a B^*– algebra .

(iii) Let ℓ^{∞} be the space of all bounded sequences . The norm on ℓ^{∞} is given by

$$\| a \| = \sup \{ | a_n | : n \in \mathbb{N} \}.$$

Let $a , b \in \ell^{\infty}$. We define

$$a b = (a_n b_n)_{n=1}^{\infty} .$$

Define involution $*$ on ℓ^{∞} by

$$a^* = (\overline{a_n})_{n=1}^{\infty} .$$

Then ℓ^{∞} is a B^* – algebra .

Theorem 4.3.1 [19]

Let A be a B^* – algebra . Then the involution on A is unique .

We state and prove some results concerning B^* – algebras .

Lemma 4.3.2

Let A be a B^* – algebra with unit $e = 1$. Then $\| 1 \| = 1$.

Proof

$$\| 1 \|^2 = \| 1^* . 1 \| = \| 1 \| \quad (\text{ since } 1^* = 1) .$$

It follows that $\| 1 \| = 1$.

Theorem 4.3.3

Let A be a B^* – algebra and $x \in A$. Then

$$\text{(i)} \quad \| x \| = \| x^* \| .$$

$$\text{(ii)} \quad \| x^* x \| = \| x^* \| \, \| x \| .$$

Proof

(i) Let $x \in A$. Then

$$\| x \|^2 = \| x^* x \| \le \| x^* \| \, \| x \| .$$

Hence $\| x \| \le \| x^* \|$.

It follows that

$$\| x^* \| \le \| x^{**} \| = \| x \| .$$

Thus $\| x \| = \| x^* \|$.

(ii) Let $x \in A$. Then

$$\| x^* x \| = \| x \|^2 . \qquad (1)$$

We have

$$\| x^* \| \| x \| = \| x \| \| x \| \qquad (\text{ By (i) })$$

$$= \| x \|^2 . \qquad (2)$$

From (1) and (2) , we obtain

$$\| x^* x \| = \| x^* \| \| x \| .$$

Lemma 4.3.4

Let A be a Banach star algebra. Let $x \in A$ such that $\| x^* \| = \| x \|$ and $\| x^* x \| = \| x^* \| \| x \|$. Then A is a B^*-algebra.

Proof

Let $x \in A$. Then

$$\| x^* x \| = \| x^* \| \| x \|$$

$$= \| x \| \| x \|$$

$$= \| x \|^2 .$$

Hence A is a B^*-algebra.

Theorem 4.3.5 [11]

Let A be a B^*-algebra. Let $x \in A$. If $x_n \to x$ in A, then $x_n^* \to x^*$.

Theorem 4.3.6

Let A be a B^*-algebra. Let x be hermitian in A. Then

(i) $r_A (x) = \| x \|$.

(ii) $r_A (x^* x) = \| x \|^2$.

(iii) $r_A (x) = r_A (x^*)$.

Proof

(i) Let x be hermitian in A. Then

$$x^* = x \ .$$

So

$$\| x^2 \| = \| x^* x \| = \| x \|^2 .$$

Since x^2 , x^4 , x^8 , ... are all hermitian , we obtain

$$\| x^4 \| = \| x^2 \|^2$$

$$= \| x \|^4 .$$

It follows that

$$\| x^{2^n} \| = \| x \|^{2^n} \quad (n = 1 , 2 , 3 , ...) \ .$$

We obtain

$$\| x^m \| = \| x \|^m \quad \text{for} \quad m = 2^n .$$

Therefore

$$r_A (x) = \lim_{m \to \infty} (\| x^m \|^{\frac{1}{m}})$$

$$= \lim_{m \to \infty} (\| x \|^{m \frac{1}{m}})$$

$$= \| x \| .$$

(ii) Let x be hermitian in A . Then $x^* x$ is also hermitian (Theorem 4.2.8) . By Theorem 4.3.6 , $r_A (x^* x) = \| x^* x \|$.

Since A is a B^*- algebra , so

$$\| x^* x \| = \| x \|^2 \ .$$

It follows that

$$r_A (x^* x) = \| x \|^2 .$$

(iii) Let x be hermitian in A . Then

$$r_A (x) = \| x \| \qquad \text{(Theorem 4.3.6) .}$$

Since x^* is hermitian (Lemma 4.2.2) , so

$$r_A (x^*) = \| x^* \| .$$

Since A is a B^*- algebra , so

$$\| x \| = \| x^* \| \qquad \text{(Theorem 4.3.3) .}$$

Hence $r_A (x) = r_A (x^*)$.

Theorem 4.3.7

Let A be a B^*- algebra . Let x be hermitian in A . Then

$$\text{(i) } r_A (x^* x) = r_A (x)^2 .$$

$$\text{(ii) } r_A (x^* x) = r_A (x^*)^2 .$$

Proof

(i) Let x be hermitian in A . Then

$$r_A (x^* x) = \| x \|^2 \qquad \text{(Theorem 4.3.6) .}$$

Since $r_A (x) = \| x \|$ (Theorem 4.3.6) , so

$$r_A (x^* x) = r_A (x)^2 .$$

(ii) The proof follows by (i) and Theorem 4.3.6 .

Definition 4.3.2

A homomorphism mapping h from a Banach star algebra A into a Banach star algebra B is called *a star homomorphism* if

$$h (x^*) = (h (x))^* \qquad (x \in A).$$

Proposition 4.3.8

Let A be a commutative Banach star algebra with unit . Let x be in the radical of A . Let ϕ be a star homomorphism . Then $r_A (x^*) = 0$.

Proof

Let $x \in \text{rad}(A)$. Then

$$\phi (x) = 0 \quad \text{for all} \quad \phi \in \phi_A \qquad \text{(Lemma 3.2.8) .}$$

$$r_A (x^*) = \sup_{\phi \in \phi_A} (| \phi (x^*) |)$$

$$= \sup_{\phi \in \phi_A} (| (\phi (x))^* |)$$

$$= 0 \qquad \left(\text{since } 0^* = 0 \text{ (Lemma 4.1.1)} \right) .$$

Theorem 4.3.9 (Gelfand - Naimark) [19]

Let A be a commutative B^*- algebra . Let $x \to \hat{x}$ be the Gelfand transform . Then

$$(x^*)^\wedge = \overline{\hat{x}} \quad (x \in A).$$

In particular , x is hermitian if and only if $\hat{x}$ is a real – valued function .

Theorem 4.3.10

Let A be a commutative Banach star algebra and $x \in A$. Then $x \to \hat{x}$ is a star homomorphism .

Proof

Let $x \in A$. Then by Theorem 4.2.9 , x has a unique representation

$$x = h + i\ k ,$$

where h , k are hermitian elements in A .

Then by Gelfand – Naimark Theorem , $\hat{h} , \hat{k}$ are real - valued functions on ϕ_A . Let $x \in A$, $\phi \in \phi_A$. Then

$$\begin{aligned} x^* (\phi) &= (h + i\ k)^* (\phi) \\ &= (h^* - i\ k^*)(\phi) \\ &= (h - i\ k)(\phi) \\ &= h^\wedge(\phi) - i\ k^\wedge(\phi) \\ &= h^*(\phi) - i\ k^*(\phi) \\ &= (h(\phi) + i\ k(\phi))^* \\ &= (x(\phi))^* . \end{aligned}$$

Theorem 4.3.11

Let A be a commutative B^*- algebra . Let $x \to \hat{x}$ be a star homomorphism . Then

$$r_A(x^* x) = r_A(x)^2 \quad (x \in A) .$$

Proof

Let $x \in A$. Then

$$r_A(x^* x) = \sup_{\phi \in \phi_A} (|(x^* x)^\wedge(\phi)|)$$

$$= \sup_{\phi \in \phi_A} (|(x^\wedge(\phi))^* x^\wedge(\phi)|)$$

$$= \sup_{\phi \in \phi_A} (|(\overline{x^\wedge}(\phi) \; x^\wedge(\phi)|)$$

$$= \sup_{\phi \in \phi_A} (|(x^\wedge(\phi)|^2)$$

$$= r_A(x)^2 .$$

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الخلاصـــــة

في هذه الرسالة سوف نناقش مفهوم جبور بنـــاخ و نعطى بعض النتائج المتعلقة في مجال جبور بناخ .

وأيضا سوف نناقش المفاهيم الآتية :

- الدوال الضربية على جبور بناخ .
- الدوال الارتدادية على جبور بناخ .
- جبور -*B .

سوف نعطى بعض النتائج والعلاقات المرتبطة بالمفاهيم السابقة .

شكر وتقدير

اعجز عن انتقاء الكلمات المناسبة التي اعبر من خلالها عن مدى
عرفاني بفضل الله سبحانه وتعالى لما وهبني إياه من نعم لاتعد ولا تحصى فالحمد لله والصلاة
والسلام على رسول الله .

أتقدم بشكـري العظيم وامتناني الكـامل إلى مشرفي الفاضل
الأستـاذ الدكـتور عبدالله خليفه البركي
لمساعدته الكبيرة وتوجيهاته الصبورة أثناء تحضير هذا البحث .
في الحقيقة هو ليس مشرف بارز ورائع فحسب لكنه أب مهتم ومتواضع.

كما أتقدم بشكري العظيم إلى أعضاء هيئة التدريس بقسم الرياضيات الذين قاموا
بتدريسي ومساعدتي خلال فترة دراستي .

كما أتقدم بخالص الشكر إلى أبله انتصار البرغثى لمساعدتها الثمينة طوال فترة الدراسه.

وأخيرا ، أتقدم بشكري العظيم والعميق إلى أبى و أمي والى أخوتي وأخواتي
و اصدقائى الذين قاموا بدعمي .

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