University of Benghazi

Faculty of Science

Department of Mathematics

Asymptotically Regular Mappings And The Fixed Point Theorems

Submitted for partial fulfillment of the requirements for the award of degree of Master of Science in Mathematics

By

Eman Ibrahim Alhamaly

Supervisor

Dr. Mohammad Din Khan

Benghazi-Libya

2011-2012

بسم الله الرحمن الرحيم

قَالُوا سُبْحَانَكَ لَا عِلْمَ لَنَا إِلَّا مَا عَلَّمْتَنَا إِنَّكَ

أَنْتَ الْعَلِيمُ الْحَكِيمُ

صدق الله العظيم

Dedicated

To

My parents

Acknowledgement

It is good fortune and matter of proud and privilege for me to have a brilliant supervisor Dr Mohammad Din Khan, Asociate Professor, Deptt of Maths. Faculty of science, Benghazi University, Benghazi , Libya .I gratefully acknowledge my indebtedness to him for his able guidance sympathetic and inspiring attitude constant encouragement and cordial cooperation for the inception to the completion of this dissertation .It is unforgettable that whenever ;I faced any problem he encouraged me to move ahead through his stimulating discussions valuable suggestions benevolent behavior and moral support . I owe him once more for all that I learnt and gained from him through his persistent help and encouragement which kept me spirited and fed me with great interest and enthusiasm of this work.

I express my sincere thanks to all the faculty members. I regard myself highly fortunate and lucky in getting enrolled in a department where serene academic atmosphere prevails created by the deep interest and entiring efforts of the Chairman, Deptt of mathematics Faculty of Science, Benghazi university , Benghazi ,Libya .

I am indebted to my parents and my brothers without whose help and encouragement , it would have not been possible for me to complete this work efficiently.

My thanks are due to all my colleagues for helping me in various ways. I also take this opportunity to express my sincere thanks to all those who have assisted me during the course of this study.

EMAN IBRAHIM ALHAMALY

CONTENTS

REFERENCES : 60 -66.

ABSTRACT

In the presentation dissertation we have discussed fixed point theorem in complete metric spaces and metrically convex spaces which generalize Banach contraction principle and the results of Kannan,Assad and Chaterjea,where the notion of asymptotically regular of maps by Petryshyn.

INTRODUCTION

Metric fixed point theory is a branch of fixed point theory which finds primary application in functional analysis . Historically speaking , the most fruitful and fundamental concept of metric spaces was introduced by French mathematician , M . Frechet in his doctorial dissertation submitted to Paris university in 1906.Metric spaces play crucial role in the further development of analysis and topology they present a natural setting for the rapid development of fixed point theory .

The study of contractive mappings played a central role in metric fixed point theory. The first metrical fixed point theorem for contractive mapping was given a Polish mathematician Stefan Banach in 1922 which is popularly known as classical Banach contraction principle. It's extensions and generalization started only at the beginning of last four decades .A good number of research papers appeared in the last four decades and by now there exist a vast literature on the subject.

The present dissertation comprises of four chapter and each chapters is divided into sections which are numbered according as they occur in the context .Each chapter begins with a brief introduction to its contents.

As usual chapter one is elementary in nature where we have discu - ssed relevant preliminary concepts ,important results and definitions which are used throughout the text. This is mainly aimed to making the dissertation as self contained as possible.

In chapter two we present some fixed point theorems for single – valued mapping which are the generalizations of well-known theorems of Banach[5],Kannan[27].

Chapter three is devoted to asymptotically regular sequences and asymptotically regular maps. The concept of asymptotically regularity is due to F. E Brouwder and W. V Petryshyn [8] .We also present the generalization of Hardy - Roger's fixed point theorem [19].

In chapter four we have incorporated the fixed point theorems for metrically convex spaces which are the generalization of Assad[(1),(2) (3)] ,Chatterjia [10] and Kannan et al [(27),(28)].

In the end , a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred to in the text.

LIST OF SYMBOLS

A=B , A ≠ B	: Equation and Inequality for sets
( )	: Diameter of a set A
d(x,y)	: Distance from one point to another
I_x	: Identity mapping on a set
Inf	: Infimum
max	: Maximum
min	: Minimum
N	: Set of natural numbers
R	: Set of real numbers
R^+	: Set of non-negative real numbers
sup	: Supremum
SoT	: S composition T
⇒	: Implies
∅	: Empty set
∈	: Belongs to, belonging to
∉	: Does not belong to
‖.‖	: Norm
| |	: Absolute value of x

CHAPTER 1

PRELIMINARIES

CHAPTER I

Preliminaries

1.1. A Brief Historical Background:

It happens quite often in mathematics that the exact solution of a system of equations can neither be determined explicitly nor it can be computed conveniently. Under such circumstances, the following question naturally arises :Does there exist any solution to the system? or a deeper question :How many different solutions the system has ? After obtaining an affirmative answer to the problem of existence , one proceeds then to look for the exact solutions. In mathematics, the problem of solving a system of equations can be reduced in general to the problem of determining the fixed points of self-mapping f of an appropriate space X. The problem of solving an equation is not only equivalent in general to the problem of determining the fixed points of a self-mapping but in fact the fixed point theory has its origin in the former.

The earliest fixed point theorem is due to L.E.J. Brouwer [7] which asserts that a continuous mapping f of the closed unit ball in R^n has at least one fixed point, that is, a point x such that $f(x)=x$. The existing literature contains various generalizations of this historic theorem. In this regard the survey article of Park [32] deserves special mention. Indeed such generalizations arise through altering the hypothesis on the space X and/or that on the mapping f itself, as suggested by the mathematical problems under investigation. For example, in the theory of differential equations and functional analysis, X is usually supposed to be a topological space of more general type, while in the theory of numerical analysis and in practical computation of fixed points, X is usually required to be compact.

Since the appearance of Brouwer's fixed point theorem in 1912 and its subsequent generalizations, fixed point theorems provided powerful tools in demonstrating the existence of solutions to a large variety of problems in applied mathematics. However, from the computational stand point, their usefulness was limited.

Brouwer's theorem was extended to infinite dimensional spaces by Schauder [39] in 1930. He proved that a continuous mapping of a compact convex subset of a Banach space has at least one fixed point. Tychonoff [41] extended Brouwer's theorem for topological vector space whereas Kakutani [30] proved a generalization of Brouwer's theorem to multi functions.

Bohnenblust and Karlin[6] gave the multi-valued analogue of Schauder's fixed point theorem whereas multi-valued analogue of Tychonoff 's fixed point theorem was given by Fan [15] and Glicksberg [16],independently.

1.2. Relevant Definitions And Results.

Definition 1.2.1.

Let X be a non - empty set . A mapping d of X x X into R (the set of reals) is said to be **a metric** (or distance function) iff d satisfies the following axioms:

[m_1] : $d(x,y) \geq 0 \quad \forall\ x, y \in X$.

[m_2] : $d(x,y) = 0$ iff $x = y$.

[m_3] : $d(x,y) = d(y,x)\ \forall\ x, y \in X$. (symmetry)

[m_4] : $d(x,y) \leq d(x,z)+d(z,y)\ \forall\ x, y, z \in X$ (triangle inequality)

If d is a metric for X, then the ordered pair (X ,d) is called a **metric space** and d(x ,y) is called the **distance** between x and y.

Example 1.2.2.

$X = R^n$; $d(x,y) = \sum (x - y)$,

For all $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R^n$,

is a metric in R^n, called the **Euclidean metric**.

Definition 1.2.3.

Let (X,d) be a metric space and let A be a non-empty subset of X. Then the **diameter** of A, denoted by δ(A), is defined by

$$\delta(A) = \sup\{ d(x,y) : x,y \in A\}$$

that is, the diameter of A is the supremum of the set of all distances between points of A.

Definition 1.2.4.

Let (X,d) be a metric space .We say that X is **bounded** if there exists a positive number M such that $d(x,y) \leq M$ for all points x and y in X.A metric space which is not bounded is said to be **unbounded** . Thus a metric space X is bounded if its diameter is finite .Similarly a subset A of X is said to be **bounded** if δ (A) is finite.

Example 1.2.5.

Let X=R and $d(x,y) = |x-y|$ .This metric space unbounded since the diameter of R is infinite.

A discrete metric space (X,d) where

$$d(x,y) = \begin{matrix} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{matrix}$$

is bounded since $\delta(X) = 1$.

Definition 1.2.6.

Let (X,d) be a metric space and let $<s_n>$ be a sequence in X. Then $<s_n>$ is said to be a **Cauchy sequence** in X if for every $\varepsilon > 0$, there exists a positive integer N such that

$$m,n \geq N \quad \Rightarrow \quad d(s_m,s_n) < \varepsilon.$$

Definition 1.2.7.

Let (X,d) be a metric space. A sequence $\{x_n\}$ in X is said to be **convergent** to a point x in X if for every $\varepsilon > 0$, there exists a positive integer n_o such that $d(x_n,x) < \varepsilon$, for all $n \geq n_o$.

Definition 1.2.8.

A metric space (X,d) is said to be **complete** if every Cauchy sequence in X converges to a point in X.

Theorem.1.2.9 [9].

Every convergent sequence is Cauchy.

Remark .

In general ,the converse of the Theorem 1.2.9 is not true.

Theorem 1.2.10 .[37].

Every Cauchy sequence is bounded.

Remark.

In general ,the converse of the Theorem 1.2.10 is not true.

Theorem 1.2.11 [37].

Every convergent sequence is bounded.

Remark.

In general ,the converse of the Theorem 1.2.11 is not true.

Definition 1.2.12.

Let (X,d) and (Y,d) be metric spaces , a function f: $X \rightarrow Y$ is called **continuous** at $x_o \in X$ if for each $>$ there exists $>$, such that $d(f(x),f(x_o)) <$, for all $d(x,x_o) <$.

A function f is called continuous on X if f is continuous at each element of X.

Definition 1.2.13.

Let X be a linear space over a filed K (real or complex) .Let $\|.\|$ be a function from X into K , such that :

(n_1) $\|x\| \geq 0$,for all $x \in X$.

(n_2) $\|x\| = 0 \iff x = 0$, the null elements of X.

(n_3) $\| \lambda x \| = |\lambda| . \| x\|$ for any $x \in X$ & any scalar λ

(n_4) $\| x+y\| \leq \|x\| + \|y\|$ for all $x,y \in X$. (The triangle inequality).

The pair (X, $\| . \|$) is called **normed (linear) space** .

Definition 1.2.14.

A subset C of a linear space X is said to be **convex** if $\alpha x + (1-\alpha)y \in C$

whenever, $x,y \in C$ and $0 \leq \alpha \leq 1$.

Example 1.2.15.

Let X=R.

Let $C=\{x:\|x\| \leq 1\}$.

Let $x,y \in C$. Then $\|x\| \leq 1$ and $\|y\| \leq 1$.

We have

$$\| \alpha x+(1-\alpha)y \| \leq \| \alpha x \|+\|(1-\alpha) y\|$$

$$= \alpha \|x\|+(1-\alpha) \|y\|$$

$$\leq \alpha +(1-\alpha)$$

$$=1$$

Thus $\alpha x+(1-\alpha) y \in C$

Hence C is a convex set.

Definition 1.2.16.

A complete normed space (X, ‖.‖) is called a **Banach space**.

Theorem 1.2.17. [9].

Every normed space (X,‖.‖) is a metric space with the metric defined by $d(x,y) = \| - \|$ (x,y ∈X).

1.3. Banach Contraction Principle And Some Of Its Generalizations.

The simplest of all the metric fixed point theorems is the contraction mapping of theorem. A number of extensions and generalizations of this celebrated theorem have been obtained in recent years. The most significant generalization of Banach Contraction Principle is due to Jungck which appeared in 1976 and the entire contents of this dissertation revolves around this theorem and is the outcome of our endeavour to improve 'commutivity' as well as contraction' conditions in Jungck's type theorems.

The present chapter is elementary in nature where we incorporate some preliminary notions along with some relevant results which will be frequently needed in our subsequent chapters. Many interesting results and definitions related to the fixed point theory could not be accomodated because of the limited size of the text.

For a comprehensive account of fixed point theory, books by Aksoyand Khamsi [1], Dugundji and Granas [12],Goebel and Kirk [17], Istratescu [21],Rus [38] and Smart [40] are of special recommendation .

Definition 1.3.1.

A topological space X is said to have the **fixed point property** if every continuous mapping T of X into itself has a fixed point i.e. there exists a point x in X such that Tx=x, x is called a fixed point of T.

Naturally, the fixed point property is a topological property it means that if X is homeomorphic to Y and X has the fixed point property then so does Y. It is worth mentioning here that a set with fixed point property should be compact and contractive. Any set lacking one of these properties will certainly leave a mapping with no fixed point. Real line, circle, torus are the examples which do not have the fixed point property while the unit interval [0,1] has the fixed point property.

In 1953, Kinoshita [31] gave an example to show that compactness and contractibility are no longer the necessary and sufficient conditions for a space to have the fixed point property. For further details one is referred to Smart [40].

The other fundamental but very simple result after Brouwer's fixed point theorem was given by Banach [5] in 1922 which is popularly known as Banach Contraction Principle or contraction mapping theorem.

Definition 1.3.2.

A mapping T from a metric space X into itself is said to be a **contraction** if

(a). . . . $d(Tx,Ty) \leq \alpha d(x,y)$, for all x,y in X and $0 \leq \alpha < 1$.

A contraction mapping is continuous but not conversely.

The Banach contraction principle states that:

(A CONTRACTION MAPPING OF A COMPLETE METRIC SPACE X INTO ITSELF HAS A UNIQUE FIXED POINT IN X).

It is the simplest of all the fixed point theorems so for established and its proof does not require much topological background. We use the contraction mapping theorem to establish the existence-uniqueness theorem for ordinary non-linear differential equations. For various other applications of the contraction mapping theorem one is referred to Kolmogrove and Fomin [29

],where one finds excellent illustrations of the use of fixed point theorems in analysis.

Mostly authors have replaced the contractive conditions by some more general mapping conditions. Rhoades ([33],[34]) has compared all contractive conditions and derived about 125 such relations.

However, we mention here a few of them .In the sequel T is a self - mapping of a metric space (X,d).

I . Edelstein[13]. For all x,y in X, $x \neq y$

(b). . . . . $d(Tx,Ty) < d(x,y)$.

This mapping is called **contractive**. A contractive mapping is continuous and has a unique fixed point if there is one. Unlike contraction mapping a contractive mapping on a complete metric space may not necessarily have a fixed point as evident from the following example:

Example 1.3.3.

The space $X=[1,\infty)$ of reals is complete.

Let T be defined by $Tx= x+1/x$, then for all x,y in X and $x< y$,

$d(Tx,Ty)= (y-x)-(1/x - 1/y) < d(x,y)$.

Thus T is contractive but T has no fixed point .

(i i) Kannan([27],[28]). Let S,T : $X \to X$. There exists α in [0,1/2) such that

(c) . . . . $d(Sx,Ty) \leq \alpha\,[d(x,Sx)+d(y,Ty)]$, for all x,y in X.

If S=T, then above condition yields.

(d). . . . $d(Tx,Ty) \leq \alpha\,[d(x,Tx)+d(y,Ty)]$, for all x,y in X.

(i i i) Reich[36] . For all x,y in X, $\alpha,\beta,\gamma \geq 0$ and $\alpha+\beta+\gamma < 1$,

(e). . . . . $d(Tx,Ty) \leq \alpha\, d(x,Tx)+\beta d(y,Ty)+\gamma d(x,y)$.

(iv) Hardy-Rogers[19]. For all x,y in X, $a_i \geq 0$ and $\sum a_i < 1$,

(f). . . .$d(Tx,Ty) \leq a_1 d(x,Tx)+a_2 d(y,Ty)+a_3 d(x,Ty)+a_4 d(y,Tx)+a_5(x,y)$.

1.4. Some Basic Fixed Point Theorems.

Definition 1.4.1.

Let T be a self mapping on a nonempty set X. A point $x \in X$ is called a **fixed point** of T if $T x = x$, i.e., a point which remains invariant under the mapping T is called a fixed point of T.

and we denote by F_T or Fix (T) the set of all fixed points of T.

Examples 1.4.2 .

(1) If $X = R$ and $T(x) = x^2+5x+4$, then $F_T = \{-2\}$;

(2) If $X = R$ and $T(x) = x^2 - x$, then $F_T = \{0,2\}$;

(3) If $X = R$ and $T(x) = x+2$, then $F_T = \emptyset$;

(4) If $X = R$ and $T(x) = x$, then $F_T = R$.

Definition 1.4.3.

Let T be a self mapping of a metric space X. Then T is said to be of **Lipschitz class** if there exists a real number $\alpha > 0$ such that $d(Tx,Ty) \leq \alpha\, d(x,y)$ for all $x,y \in X$. If $\alpha<1$, then T is called **contraction map**. In case $d(Tx,Ty) < d(x,y)$, $x \neq y$, then T is said to be a **contractive map**.

Example 1.4.4.

Define $T: R \rightarrow R$ by

$$T(x) = -x + 1.$$

With the usual metric $d(x,y) = |x - y|$.

Let $x, y \in X$. Then

$$|T(x) - T(y)| = \quad -x + 1 \quad - \quad -y + 1$$

$\leq | - x - - y|$

$= - |x - y|$.

Thus T is a contraction mapping with $\alpha = -$ and $x = -$ is a fixed point of T.

Definition 1.4.5.

Let T_1 and T_2 be two functions from a non - empty set X into itself . If there exists an element x in X such that

$$T_1(x)=T_2(x)=x,$$

Then x is called a **common fixed point** of T_1 and T_2.

Similarly, if $n \in N$ and $T_1, T_2, \ldots, T_n$ are functions from X into itself.

If there exists an element x in X such that

$$T_1(x)=T_2(x)= \ldots =T_n(x) =x,$$

Then x is called a **common fixed point** of $T_1, T_2, \ldots, T_n$.

Example 1.4.6.

Define $T_1, T_2: R \rightarrow R$ by

$$T_1(x)=2x-2 , \quad T_2(x)=x^2-3x+4.$$

Then $x=2$ is a common fixed point of T_1, T_2.

Definition 1.4.7.

Let (X,d) be metric space the mappings $f,g: X \to X$ are said to be **commute** iff $fg = gf$.

1.5. The iteration process.

In an iteration process, we choose an arbitrary point x_o in a given set and calculate recursively a sequence $\{x_o, x_1, x_2, ...\}$ from a relation of the form $x_{n+1} = T x_n$,i.e., for arbitrary x_o one successively writes $x_1 = T x_o$, $x_2 = Tx_1 = T^2 x_o$, ...

With the development of fast computer, iteration schemes are used in nearly every branch of applied mathematics and convergence proofs and error estimates are very often obtained using some fixed point theorems.

We begin with the following results which deal with the Convergence of the sequence of iterates for continuous functions defined on closed interval. The following result is due to Hillam [20].

For a given self map the following properties obviously hold:

1) $F_T \subset F_T^{\,n}$, for each $n \in N$;

2) $F_T^{\,n} = \{x\}$, for some $n \in N \Rightarrow F_T = \{x\}$;

The reverse of (2) is not true, in general, as shown by the following example.

Example 1.5.1.

Let $T:\{1,2,3\} \rightarrow \{1,2,3\}$, $T(1)=3, T(2)=2$ and $T(3)=1$.

Then $F_{T^2}=\{1,2,3\}$ but $F_T=\{2\}$.

CHAPTER 2

FIXED POINT THEOREMS IN COMPLETE METRIC SPACES

CHAPTER II

FIXED POINT THEOREMS IN COMPLETE METRIC SPACE.

2.1. **INTRODUCTION**.

In the present chapter we present some fixed point theorems for single –valued mapping . Our results in section 2.2 are the generalizations of the well –known theorems of Banach [5] and Kannan [27].

2 . 2 . **SINGLE – VALUED MAPS AND THEIR FIXED POINTS**.

Throughout this section (X,d) stands for a complete metric space .

Theorem 2.2.1.

If T is a self mapping of (X,d) and if for some positive integer p ,the following inequality :

(A)....$d(T^{2p}x,T^{2p}y)\leq a_1 d(T^p x,T^{2p}x)+a_2 d(T^p y,T^{2p}y)+a_3 d(T^p x,T^p y)$,

holds for all x,y in X,$a_1\geq0,a_2\geq0,a_3\geq0$ and $a_1+a_2+a_3<1$, then T has a unique fixed point provided T^p is continuous.

Proof.

Let $x \in X$. Set $T^p(x)=x_0$ and $T^p(x_{n-1})=x_n$, for $n=1,2,3,\ldots$

Also suppose K= ——

Then ,

$$d(x_1,x_2)=d(T^{2p}x,T^{2p}x_0) \leq a_1d(T^px,T^{2p}x)+a_2d(T^px_0,T^{2p}x_0)+a_3d(T^px,T^px_0)$$

$$\leq a_1d(x_0,x_1)+a_2d(x_1,x_2)+a_3d(x_0,x_1).$$

Hence

$$d(x_1,x_2) \leq k\, d(x_0,x_1).$$

Again

$$d(x_2,x_3)=d(T^{2p}x_0,T^{2p}x_1) \leq a_1d(T^px_0,T^{2p}x_0)+a_2d(T^px_1,T^{2p}x_1)+a_3d(T^px_0,T^px_1),$$

$$\leq a_1d(x_1,x_2)+a_2d(x_2,x_3)+a_3d(x_1,x_2).$$

So

$$d(x_2,x_3) \leq k^2d(x_0,x_1).$$

In general , we have

$$d(x_n,x_{n+1}) \leq k^nd(x_0,x_1).$$

Thus $\{ x_n \}$ is a Cauchy sequence which converges to some $w \in X$. Now we consider the inequality

$$d(w,T^{2p}w) \leq d(w,x_{n+2})+d(T^{2p}x_n,T^{2p}w)$$

$$\leq d(w,x_{n+2})+a_1d(T^px_n,T^{2p}x_n)$$

$$+a_2d(T^pw,T^{2p}w)+a_3d(T^px_n ,T^pw)$$

$$\leq d(w,x_{n+2})+ a_1d(x_{n+1},x_{n+2})$$

$$+ a_2d(T^pw,x_{n+1})+a_2d(x_{n+1},x_{n+2})$$

$$+a_2d(x_{n+2},T^{2p}w)+a_3d(x_{n+1},T^pw).$$

The right hand side of the above inequality can be made arbitrarily small by choosing n sufficiently large .

Hence $T^{2p}(w)=w$.

For uniqueness of w , let w^* be another fixed point of T^{2p}.Then

$$d(w,w^*)=d(T^{4p}w,T^{4p}w^*)$$

$$\leq a_1d(T^{2p}w,T^{4p}w)+a_2d(T^{2p}w^*,T^{4p}w^*)+a_3d(T^{2p}w,T^{2p}w^*)$$

$$=a_1d(T^{2p}w,T^{4p}w)+a_2\, d(T^{2p}w^*,T^{4p}w^*)+a_3d(T^{4p}w,T^{4p}w^*).$$

So

$(1-a_3)\, d(T^{4p}w, T^{4p}w^*) \leq 0.$

Therefore w is the unique fixed point of T^{2p} and hence a unique fixed point of T.

This completes the proof.

Theorem 2.2.2.

Let T be a self-mapping of X satisfying

(B)….$d(T^{p+1}x, T^{p+1}y) \leq a_1 d(T^p x, T^{p+1}x) + a_2 d(T^p y, T^{p+1}y) + a_3 d(T^p x, T^{p+1}y)$

$+ a_4 d(T^p y, T^{p+1}x) + a_5 d(T^p x, T^p y)$, for all $x, y \in X$,

(C) …. $\sum_i^5 a_i < 1$, $a_i \geq 0$, $1 \leq i \leq 5$,

(D) …. T^p is continuous.

Then T has a unique fixed point .

Proof.

For an arbitrary $x \in X$, define $x_0 = T^p(x)$ and $x_n = T(x_{n-1})$. Then

$$d(x_1, x_2) = d(T^{p+1}x, T^{p+2}x)$$

$$\leq a_1 d(T^p x, T^{p+1}x) + a_2 d(T^{p+1}x, T^{p+2}x) + a_3 d(T^p x, T^{p+2}x)$$

$$+ a_4 d(T^{p+1}x, T^{p+1}x) + a_5 d(T^p x, T^{p+1}x)$$

$\leq a_1d(x_0,x_1)+a_2d(x_1,x_2)+a_3d(x_0,x_2)+a_4d(x_1,x_1)$

$+a_5d\,(x_0,x_1).$

Hence,

$d(x_1,x_2) \leq (\text{———————})\; d(x_0,x_1) = k\; d(x_0,x_1),$

where $\quad k = (\text{———————})$

Further, we have

$d(x_2,x_3)=d(T^{p+2}x,T^{p+3}x) \leq a_1d(T^{p+1}x,T^{p+2}x)+a_2d(T^{p+2}x,T^{p+3}x)$

$+a_3d(T^{p+1}x,T^{p+3}x)+a_4d(T^{p+2}x,T^{p+2}x)$

$+a_5d(T^{p+1}x,T^{p+2})$

$\leq a_1d(x_1,x_2)+a_2d(x_2,x_3)+a_3d(x_1,x_3)$

$+a_4d(x_2,x_2)+a_5d(x_1,x_2).$

Hence

$d(x_2,x_3) \leq k\; d(x_1,x_2) \leq k^2d(x_0,x_1).$

In general ,

$d(x_n,x_{n+1}) \leq k^n\; d(x_0,x_1).$

Thus $\{ x_n \}$ is a Cauchy sequence which converges to some point w in X.

Now,

$$d(T^p w, T^{p+1} w) \leq d(T^p w, x_{n+p+1}) + d(T^{p+1} x_n, T^{p+1} w)$$

$$\leq d(T^p w, x_{n+p+1}) + a_1 d(T^p x_n, T^{p+1} x_n) + a_2 d(T^p w, T^{p+1} w)$$

$$+ a_3 d\,(T^p x_n, T^{p+1} w) + a_4 d(T^{p+1} x_n, T^p w) + a_5 d(T^p x_n, T^p w)$$

$$\leq d(T^p w, x_{n+p+1}) + a_1 d(x_{n+p}, x_{n+p+1}) + a_2 d(T^p w, w)$$

$$+ a_2 d(w, T^{p+1} w) + a_3 d(x_{n+p}, w) + a_3 d(w, T^{p+1} w)$$

$$+ a_4 d(x_{n+p+1}, T^p w) + a_5 d(x_{n+p}, T^p w).$$

Hence , we have

$$(1-a_2-a_3) d(T^p w, T^{p+1} w) \leq d(T^p w, x_{n+p+1},) + a_1 d(x_{n+p}, x_{n+p+1})$$

$$+ a_2 d(T^p w, w) + a_3 d(x_{n+p}, w)$$

$$+ a_4 d(x_{n+p+1}, T^p w)$$

$$+ a_5 d(x_{n+p}, T^p w).$$

Using the continuity of T^p and letting $n \to \infty$,we get $T^p(w) = T^{p+1}(w)$.

Hence $T^p(w)$ is a fixed point of T. For the unicity of $T^p(w)$, consider

another fixed point $w^* \neq T^p w$ of T.

Then we get,

$$d(T^p w,w^*)=d(T^p w,T^{p+1}w^*)$$

$$=d(T^{p+1}w,T^{p+1}w^*)$$

$$\leq a_1 d(T^p w,T^{p+1}w)+a_2 d(T^p w^*,T^{p+1}w^*)$$

$$+a_3 d(T^p w,T^{p+1}w^*)+a_4 d(T^p w^*,T^{p+1}w)+a_5 d(T^p w,T^p w^*)$$

$$= a_3 d(T^p w,T^{p+1}w^*)+a_4 d(T^{p+1}w^*,T^p w) +a_5 d(T^p w,T^{p+1}w^*).$$

This gives $d(T^p w,T^{p+1}w^*)=d(T^p w,w^*)=0$.

Therefore $T^p(w)$ is the unique fixed point of T .

This completes the proof.

Theorem 2.2.3.

If T be a mapping of X into itself with d as metric and if

$$d\,(Tx,Ty) \leq \frac{(\;,\;).\;(\;,\;).\;(\;,\;)}{(\;,\;)\;(\;,\;)\quad(\;,\;)\;(\;,\;)\quad(\;,\;)\;(\;,\;)}$$

for $1/3 < \beta < 1$ for all $x,y \in X$.

Then T have a unique fixed point.

Proof.

We define a sequence $\{ x_n\}$ of elements of X as follows :

Let $x_0 \in X$ and $x_n = Tx_{n-1}$, for n is positive integer ,then

$$d(x_1,x_2)=d(Tx_0,Tx_1)$$

$$\le \frac{(\ ,\ )\ (\ ,\ )\ (\ ,\ )}{(\ \ ,\ )\ (\ ,\ )\quad (\ ,\ )\ (\ ,\ )\quad (\ ,\ )\ (\ ,\ )]}$$

Or $2d(x_1,x_2)+d(x_0,x_1) \le 3\beta d(x_0,x_1)$

Or $d(x_1,x_2) \le 3\beta\text{-}1 / 2 \ \ d(x_0,x_1)$.

Similarly

$$d(x_2,x_3) \le \frac{\quad}{\quad} \ d(x_1,x_2)$$

Or $d(x_2,x_3) \le (\frac{\quad}{\quad})^2 \ d(x_0,x_1)$.

In general

$$d(x_n,x_{n+1}) \le (\frac{\quad}{\quad})^n \ d(x_0,x_1).$$

There for

$$d(x_n,x_{n+p}) \le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2}) +. . . .+d(x_{n+p-1},x_{n+p})$$

$$\leq \frac{}{} d(x_0,x_1)$$

Where $\lambda = \frac{}{}$.

Hence $\{ x_n\}$ is a Cauchy sequence .

Since X is complete the sequence will converge to $x_0 \in X$.

Now we shall show that $x_0 = Tx_0$. We write

$$d(x_0,Tx_0) \leq d(x_0,x_n) + d(Tx_{n-1},Tx_0)$$

$$\leq d(x_o,x_n) + 3\beta[d(x_{n-1},x_o)+d(x_o,x_n)]d(x_o,x_1)d(x_{n-1},x_o)/$$

$$[d(x_{n-1},x_n)d(x_o,x_1)+d(x_o,x_1).d(x_{n-1},x_o) + d(x_{n-1},x_o)d(x_{n-1},x_n)]$$

Taking limit $x_n \rightarrow x_0$ As $n \rightarrow \infty$

$$d(x_0,Tx_0) \leq 0.$$

Hence x_0 is the fixed point of T.

Now to show the uniqueness of x_0 . Let us consider y_0 be any other fixed point of T , then

$$d(x_0,y_0) = d(Tx_o,Ty_o)$$

$$\leq 3\beta\, d(x_o,Tx_o).d(y_o,Ty_o).d(x_0,y_o) / [\, d(x_o,Tx_o)\, d(y_o,Ty_o)$$

$+ \; d(y_o,Ty_o)\, d(x_o,y_o)+d(x_o,y_o)d(x_o,Tx_o)]$

or $d(x_o,y_o) \leq 0$.

Hence x_o is the unique fixed point of T.

This complete the proof.

Remark:

For p=0 ,Theorem 2.2.2 reduces to that of Hardy and Rogers [19].

CHAPTER 3

ASYMPTOTICALLY REGULSEQUENCES AND MAPS

CHAPTER III

ASYMPTOTICALLY REGULAR SEQUENCES AND MAPS

3.1. INTRODUCTION.

In recent years, a number of generalizations of asymptotically T-regular sequences and asymptotically regular maps have been discussed by many authors. The concept of asymptotic regularity is due to F.E.Browder and W.V .Petryshyn [8]. The work of Engl [14],Reich [35] and Gornicki [18] are of special significance. In this chapter we present some generalizations of hardy –Roger's fixed point theorem [19] for certain sequences and mappings which are asymptotically regular . The technique of our proofs can be used equally well for other results in literature to show that one need not consider the sequence of successive approximation to prove the existence of fixed points of contraction mappings.

3.2.Results For Asymptotically Regular Sequence:

The following definition is essentially borrowed from Engl [14].

Definition 3.2.1.

Let (X ,d) be a metric space . A sequence $\{ x_n \}$ in X is said to be **asymptotically T-regular** if $\lim_{n\to\infty} d(x_n ,Tx_n)=0$.

Theorem 3.2.2.

Let (X ,d) be a complete metric space and T a self –mapping of X satisfying the inequality:

(B) $d(Tx,Ty) \leq a_1d(x,Tx) + a_2d(y,Ty) + a_3d(x,Ty) + a_4d(y,Tx) +a_5d(x,y)$,

for all $x,y \in X$, where a_i (i=1,2,3,4,5) are non- negative reals and $\max\{(a_1+a_4), (a_3+a_4+a_5)\}<1$. If there exists an asymptotically T –regular sequence in X, then T has a unique fixed point.

Proof.

Let $\{x_n\}$ be an asymptotically T-regular sequence in X. Then

$$\begin{aligned}
d(x_n,x_m) &\leq d(x_n,Tx_n) + d(Tx_n,x_m) \\
&\leq d(x_n,Tx_n)+d(Tx_n,Tx_m)+d(Tx_m,x_m) \\
&\leq d(x_n,Tx_n)+d(Tx_m,x_m)+\{a_1d(x_n,Tx_n) +a_2d(x_m,Tx_m) \\
&\quad +a_3d(x_n,Tx_m) + a_4d(x_m,Tx_n) + a_5d(x_n,x_m)\} \\
&\leq d(x_n,Tx_n)+d(Tx_m,x_m)+a_1d(x_n,Tx_n) + a_2 d(x_m,Tx_m) \\
&\quad + a_3 d(x_n,x_m) + a_3 d(x_m,Tx_m) + a_4 d(x_m,x_n)+ a_4 d(x_n,Tx_n)
\end{aligned}$$

$+ a_5 d(x_n,x_m)$.

Thus ,we get

$$d(x_n,x_m) \leq \{(1+a_1+a_4)/(1-a_3-a_4-a_5)\} d(x_n,Tx_n)$$

$$+ \{(1+a_2+a_3)/(1-a_3-a_4-a_5)\} d(x_m,Tx_m).$$

Taking limit as n tends to infinity, we have $\lim_{n\to\infty} d(x_n,x_m)=0$, showing thereby that $\{x_n\}$ is a Cauchy sequence. Since X is complete ,put $\lim_{n\to\infty} x_n=z$ (say).

Now we claim that z is a fixed point of T consider,

$$d(Tz,z)\leq d(Tz,Tx_n) + d(Tx_n,x_n) + d(x_n,z)$$

$$\leq a_1 d(z,Tz)+ a_2 d(x_n,Tx_n)+ a_3 d(z,Tx_n)+ a_4 d(x_n,Tz)$$

$$+ a_5 d(z,x_n)+ d(Tx_n,x_n) + d(x_n,z).$$

$$\leq a_1 d(z,Tz)+ a_2 d(x_n,Tx_n)+ a_3 d(z,x_n)+ a_3 d(x_n,Tx_n)+ a_4 d(x_n,z)$$

$$+ a_4 d(z,Tz)+ a_5 d(z,x_n)+ d(Tx_n,x_n)+ d(x_n,z).$$

Therefore,

$$(1-a_1-a_4)d(Tz,z) \leq (1+a_2+a_3)d(x_n,Tx_n)+(1+a_3+a_4+a_5)d(x_n,z),$$

which gives

$$d(Tz,z) \leq \{(1+a_2+a_3) / (1-a_1-a_4)\} d(x_n,Tx_n)$$

$$+ \{(1+a_3+a_4+a_5) / (1-a_1-a_4)\} d(x_n,z).$$

Since T is asymptotically T-regular, taking limit as $n\to\infty$, we are left with

$d(Tz,z)=o$, i.e. $T z=z$

Hence z is a fixed point T.

To show the uniqueness, let $z\neq z_1$ be two fixed points, then

$$d(z,z_1)=d(Tz,Tz_1)$$

$$\leq a_1d(z,Tz)+a_2d(z_1,Tz_1)+a_3d(z,Tz_1)+a_4d(z_1,Tz)+a_5d(z,z_1)$$

$(1-a_3-a_4-a_5)d(z,z_1) \leq 0$, since $(1-a_3-a_4-a_5) >0$,we have $d(z,z_1)=0$,so $z=z_1$

whence uniqueness follows immediately.

This completes the proof.

If T is continuous , then existence part follows very easily as shown

by the following theorem.

In this case condition (B) is not needed.

Theorem 3.2.3.

Let (X ,d) be a metric space and T a continuous self-mapping of X. If there exists an asymptotically T-regular sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n =z$, then z is a fixed point T.

Proof.

consider, $d(Tz,z) \leq d(Tz,Tx_n)+d(Tx_n,x_n)+d(x_n,z)$.

Then taking limit as $n\to\infty$, we have

$$d(Tz,z)=0,$$

so $$Tz=z.$$

Hence z is a fixed point of T.

3.3.Results for Asymptotically Regular Maps:

Following Browder and Petryshyn [8],we have the following:

Definition 3.3.1.

Let (X ,d) be a metric space .A mapping T of X into itself is said to be **asymptotically regular** at a point x in X if $\lim_{n\to\infty} d(T^n x, T^{n+1}x)=0$.

Theorem 3.3.2.

Let (X ,d) be a complete metric space and T a self –mapping of X satisfying the inequality:

$$d(Tx,Ty) \leq a_1 d(x,Tx)+a_2 d(y,Ty)+a_3 d(x,Ty)+a_4 d(y,Tx)+a_5 d(x,y),$$

for all $x,y \in X$,where $a_i(i=1,2,3,4,5)$ are non–negative reals with $\max\{(a_1+a_4),(a_3+a_4+a_5)\}<1$.If T is asymptotically regular at some point x of X ,then there exists a unique fixed point of T.

Proof.

Let T be an asymptotically regular at $x_o \in X$. Consider the sequence $\{T^n x_0\}$,then for all $m,n \geq 1$

$$d(T^m x_0,T^n x_0) \leq a_1 d(T^{m-1}x_0,T^m x_0)+a_2 d(T^{n-1}x_0,T^n x_0)+a_3 d(T^{m-1}x_0,T^n x_0)$$

$+a_4d(T^{n-1}x_0,T^mx_0)+a_5d(T^{m-1}x_0,T^{n-1}x_0)$

$\leq a_1d(T^{m-1}x_0,T^mx_0)+a_2d(T^{n-1}x_0,T^nx_0)+a_3d(T^{m-1}x_0,T^mx_0)$

$+a_3d(T^mx_0,T^nx_0)+a_4d(T^{n-1}x_0,T^nx_0)+a_4d(T^nx_0,T^mx_0)$

$+a_5d(T^{m-1}x_0,T^mx_0)+a_5d(T^mx_0,T^nx_0)+a_5d(T^nx_0,T^{n-1}x_0)$.

Hence, we get

$d(T^mx_0,T^nx_0) \leq \{(a_1+a_3+a_5) / (1-a_3-a_4-a_5)\}\ d(T^{m-1}x_0,T^mx_o)$

$+ \{(a_2+a_4+a_5) / (1-a_3-a_4-a_5)\}\ d(T^{n-1}x_0,T^nx_0)$.

Since T is asymptotically regular and $m,n\to\infty$, above yields

$\lim_{n\to\infty} d(T^mx_0,T^nx_0)=0$.

This shows that $\{T^nx_0\}$ is a Cauchy sequence, since X is complete,

$\lim_{n\to\infty} T^nx_0=z$.

Now we claim that z is fixed point of T. For this we consider,

$d(Tz,z) \leq d(Tz,T^nx_0) + d(T^nx_0,z)$

$\leq a_1d(z,Tz)+ a_2d(T^{n-1}x_0,T^nx_0)+ a_3d(z,T^nx_0)+ a_4d(T^{n-1}x_0,Tz)$

$+ a_5d(z,T^{n-1}x_0) + d(T^nx_0,z)$

$\leq a_1d(z,Tz)+ a_2d(T^{n-1}x_0,T^nx_0)+ a_3d(z,T^nx_0)+ a_4d(T^{n-1}x_0,T^nx_0)$

$+ a_4d(T^nx_0,Tz)+ a_5d(z,T^nx_0)+ a_5d(T^nx_0,T^{n-1}x_0)+ d(T^nx_0,z)$.

Letting n tending to infinity, we get

$d(Tz,z) \leq a_1d(z,Tz)+ a_4d(z,Tz)$,

$(1-a_1-a_4)d(z,Tz) \leq 0.$

Hence $Tz = z$.

Therefore z is a fixed point of T.

The unicity of fixed point z follows from Theorem 3. 2.2.

Theorem 3.3.3.

Let (X ,d) be a metric space and T a self – mapping satisfying the inequality:

$$d(Tx,Ty) \leq a_1 d(x ,Tx)+ a_2 d(y, Ty)+ a_3 d(x, Ty) + a_4 d(y, Tx)+ a_5 d(x,y)$$

for all $x ,y \in X$, where a_i (i=1,2,3,4,5) are non –negative reals with $\max\{(a_2+a_3),(a_3+a_4+a_5)\}<1$. If T is asymptotically regular at some point x in X and the sequence $\{T^n x\}$ of iterates has a subsequence { T } converging to a point z of X, then z is a unique fixed point of T and $\{T^n x\}$ also converges to z.

Proof.

Let $\lim_{k\to\infty}$ $x = z \neq Tz$, then

$$d(Tz ,z) \leq d(z, T\ \ x) +d(T\ \ x, T\ \ x)+ d(T\ \ x,Tz)$$

$$\leq d(z, T\ \ x)+ d(T\ \ x, T\ \ x)+ a_1 d(T\ \ x, T\ \ x)+ a_2 d(z ,Tz)$$

$$+\ a_3 d(T\ \ x,Tz)+ a_4 d(z, T\ \ x)+ a_5 d(T\ \ x, z).$$

Then $n \rightarrow \infty$,yields that

$d(z,Tz) \leq (a_2+a_3)d(z,Tz),$

$(1- a_2-a_3)d(z,Tz) \leq 0$,since$(1- a_2-a_3) >0$,so $Tz=z$.

whence z is a fixed point of T. Now ,

$d(z,T^nx)=d(Tz,T^nx)$

$\leq a_1 d(z,Tz)+a_2 d(T^{n-1}x,T^nx)+a_3 d(z ,T^nx)+a_4 d(T^{n-1}x,Tz)+a_5 d(z,T^{n-1}x)$

$\leq a_1 d(z,Tz)+a_2 d(T^{n-1}x,T^nx)+a_3 d(z,T^nx)+a_4 d(T^{n-1}x,T^nx)$

$+a_4 d(T^nx,Tz)+a_5 d(z,T^nx)+a_5 d(T^nx,T^{n-1}x).$

So,

$(1-a_3-a_4 -a_5) d(z, T^nx) \leq (a_2+a_4+a_5) d(T^{n-1}x,T^nx).$

$d(z, T^nx) \leq$ ———— $d(T^{n-1}x,T^nx)$

Asymptotically regularity of T at x and the fact that $(a_3+a_4+a_5)<1$,imply that the sequence $\{T^nx\}$ converges to z .

This completes the proof.

Remark.

It is clear that the asymptotic regularity of the mapping T satisfying Hardy – Roger's contraction condition is actually a consequence of $\sum a < 1$ .So our Theorem 3.3.2 and Theorem 3.3.3 extend results due to Hardy – Rogers [19].

3.4. Composite Asymptotic Regularity And Common Fixed Points:

In this section, we present a generalization to the concept of asymptotically regular (abbreviated as a.r) mapping (c.f. Definition 3.3.1) by introducing the notion of compositely asymptotically regular (abbreviated as c.a.r.) mappings.

In doing so, we are motivated by those functions which are not a.r but their composition is a.r. To substantiate this , let us consider the following example.

Example 3.4.1.

Let X=R be the set of reals equipped with usual metric. On X define the pair of maps (S,T) by

$$S(x)=x-1 \quad \text{and} \quad T(x)=x+1,$$

For all $x \in R$ .Then

$$\lim_{n\to\infty} d(S^n x,S^{n+1}x) = \lim_{n\to\infty} |x-n-x+(n+1)| =1,$$

whereas

$$\lim_{n\to\infty} d(T^n x,T^{n+1}x)= \lim_{n\to\infty} |x+n-x-(n+1)| =1,$$

which shows that both the maps S and T are not a.r. But on taking their composition , we get STx=x and hence, we deduce

$$\lim_{n\to\infty} d((ST)^n x,(ST)^{n+1}x)=\lim_{n\to\infty}|\ x-x\ |=0\ ,$$

which shows that the pair(S,T) is c.a.r. Thus it seems worthwhile to introduce the following:

Definition 3.4.2.

A pair of self-mapping (S,T) of a metric space (X, d) is said to be **compositely asymptotically regular** (abbreviated c.a.r) at a point x if their composition SoT is a.r. at x . It is immediate to note that if we choose $T=I_x$ (or $S=I_x$)**,**where I_x is the identity mapping on X, then the notion of c.a.r. mappings reduces to that of a.r. mapping.

Let R^+ denotes the set of non-negative real numbers, and let $F:R^+\to R^+$ be a mapping such that F(0) = 0 and F is continuous at 0 .Employing the notion of c.a.r. mappings, we first prove the following:

Theorem 3.4.3.

Let S and T be self – mappings of a complete metric space (X ,d) satisfying:

$$d(STx,STy) \leq \alpha \max \{d(x,y),d(x,STx),d(y,STy)\ ,d(x,STy)\ ,d(y,STx)\}$$

$$+F(\min\{d^2(x,y), d(x,STx).d(y,STy), d(x,STx).d(x,STy),$$

$$d(y,STx).d(y,STy), d(x,STy).d(y,STx)\}), \qquad (3.4.3.1)$$

for all x,y in X ,where $0 \le \alpha < 1$. Then ST has a unique fixed point z provided the pair (S,T) is c.a.r. at some point of X .Moreover , if the pair (S,T) commute at z and Tz, then the fixed point of ST also remains the fixed point of S and T separately.

Proof.

Let the pair (S,T) be c.a.r. at x_o in X, then using (3.4.3.1), we get

$$d((ST)^m x_o,(ST)^n x_o) \le \alpha \max\{d((ST)^{m-1}x_o,(ST)^{n-1}x_o),$$

$$d((ST)^{m-1}x_o,(ST)^m x_o), d((ST)^{n-1}x_o,(ST)^n x_o),$$

$$d((ST)^{m-1}x_o,(ST)^n x_o), d((ST)^{n-1}x_o,(ST)^m x_o)\}$$

$$+F(\min\{d^2((ST)^{m-1}x_o,(ST)^{n-1}x_o),$$

$$d((ST)^{m-1}x_o,(ST)^m x_o). d((ST)^{n-1}x_o,(ST)^n x_o),$$

$$d((ST)^{m-1}x_o,(ST)^m x_o). d((ST)^{m-1}x_o,(ST)^n x_o),$$

$$d((ST)^{n-1}x_o,(ST)^m x_o). d((ST)^{n-1}x_o,(ST)^n x_o),$$

$$d((ST)^{m-1}x_o,(ST)^n x_o).d((ST)^{n-1}x_o,(ST)^m x_o)\}) \qquad (3.4.3.2)$$

Substituting

$$d((ST)^{m-1}x_o,(ST)^{n-1}x_o) \le d((ST)^{m-1}x_o,(ST)^{m}x_o)+ d((ST)^{m}x_o,(ST)^{n}x_o)$$

$$+d((ST)^{n}x_o,(ST)^{n-1}x_o),$$

$$d((ST)^{m-1}x_o,(ST)^{n}x_o) \le d((ST)^{m-1}x_o,(ST)^{m}x_o)+ d((ST)^{m}x_o,(ST)^{n}x_o),$$

$$d((ST)^{n-1}x_o,(ST)^{m}x_o) \le d((ST)^{n-1}x_o,(ST)^{n}x_o) + d((ST)^{n}x_o,(ST)^{m}x_o),$$

in (3.4.3.2) and using the composite asymptotic regularity of the pair

(S,T) at x_o, we get as $n,m\to\infty$

$$d((ST)^{m}x_o,(ST)^{n}x_o) \le \alpha \max \{d((ST)^{m}x_o,(ST)^{n}x_o),0,0,$$

$$d((ST)^{m}x_o,(ST)^{n}x_o),\ d((ST)^{m}x_o,(ST)^{n}x_o)\}$$

$$+F(\min\{d^2((ST)^{m}x_o,(ST)^{n}x_o),0,0,0,$$

$$d^2((ST)^{m}x_o,(ST)^{n}x_o)\}),$$

Which yields to $d((ST)^{m}x_o,(ST)^{n}x_o) \le \alpha\, d((ST)^{m}x_o,(ST)^{n}x_o) +F(0)$.

Since $\alpha < 1$ and $F(0)=0$,we get $d((ST)^{m}x_o,(ST)^{n}x_o) \to 0$ as $n,m\to\infty$. Hence $\{(ST)^{n}x_o\}$ is a Cauchy sequence in X and so, since (X ,d) is complete ,it converges to a point z in X. Now using (3.4.3.1), we obtain

$$d(z,STz) \le d(z,(ST)^{n}x_o)+d((ST)^{n}x_o,STz)$$

$\leq d\ (z,(ST)^n x_o)+\alpha \max \{ d\ ((ST)^{n-1}x_o,z), d\ ((ST)^{n-1}x_o,(ST)^n x_o),$

$d\ (z,STz), d\ ((ST)^{n-1}x_o,STz), d\ (z,(ST)^n x_o)\}$

$+F(\min \{ d^2((ST)^{n-1}x_o,z), d\ ((ST)^{n-1}x_o,(ST)^n x_o.$

$d\ (z,STz)\ , d((ST)^{n-1}x_o,(ST)^n x_o).\ d\ ((ST)^{n-1}x_o,STz),$

$d\ (z,(ST)^n x_o).\ d\ (z,STz), d\ ((ST)^{n-1}x_o,STz).d\ (z,(ST)^n x_o)\})$

$\leq d\ (z,(ST)^n x_o)+ \alpha \max \{d\ (ST)^{n-1}x_o,z),\ d(\ (ST)^{n-1}x_o,(ST)^n x_o),$

$d\ (z,STz), [d\ (\ (ST)^{n-1}x_o,z)+d\ (z,STz)], d\ (z,(ST)^n x_o)\}$

$+F\ (\min\{ d^2(\ (ST)^{n-1}x_o,z), d((ST)^{n-1}x_o,(ST)^n x_o).d(z,STz),$

$d\ ((ST)^{n-1}x_o,(ST)^n x_o).\ d\ ((ST)^{n-1}x_o,STz), d\ (z,(ST)^n x_o).d\ (z,STz),$

$[d\ ((ST)^{n-1}x_o,z)+d(z,STz)].\ d(z,(ST)^n x_o)\}),$

which on letting $n,m \to \infty$,reduces to

$d\ (z,STz) \leq \alpha\ d(z,STz) < d\ (z,STz),$

a contradiction giving thereby $z=STz$.

The uniqueness assertion follows immediately from contraction Condition (3.4.3.1) of the hypothesis.

Now, it remains to show that z is also a common fixed point of S and T separately. For this let us write

$$Sz=S(STz)=S(TSz)=ST(Sz),$$

$$Tz=T(STz)=TS(Tz)=ST(Tz),$$

which show that Sz and Tz are other fixed points of ST.

Therefor , in view of the uniqueness of the fixed point of ST, one can write

$$Sz=Tz=STz=z$$

which shows that z is the common fixed point of S,T and ST.

Theorem 3.4.4.

Let (X,d) be a metric space and S and T be mappings of X into itself satisfying (3.4.2.1), where $0\leq \alpha <1$. If the pair (S,T) is c.a.r. at a point x in X and the sequence of iterates $\{(ST)^n\}$ has a subsequence converging to a point u in X, then u is the unique fixed point of ST and $\{(ST)^n x\}$ also converges to u. Moreover, if the pair (S,T) commutes at u and Tu, then the fixed point of ST also remains the fixed point of S and T separately.

Proof.

Let the pair (S,T) be c.a.r. at some point x of X and consider the sequence $\{ (ST)^n x\}$. Suppose that $\lim_k (\quad) x = u$ and $STu \neq u$. By (3.4.3.1), we obtain

$$d(u, STu) \leq d(u, (ST) x) + d((ST) x, (ST) x) + d((ST) x, STu)$$

$$\leq d(u, (ST) x) + d((ST) x, (ST) x)$$

$$+ \alpha \max \{ d((ST) x, u), d((ST) x, (ST) x), d(u, STu),$$

$$d(((ST) x, STu), d(u, (ST) x)\}$$

$$+F(\min\{ d^2((ST) x, u), d((ST) x, (ST) x).d(u, STu),$$

$$d((ST) x, (ST)^{nk+1} x).d((ST) x, STu), d(u, (ST) x).$$

$$d(u, STu), d((ST) x, STu).d(u, (ST) x)\}),$$

which on letting $k \rightarrow \infty$, reduces to

$d(u, STu) \leq \alpha \max \{0,0,d(u, STu),d(u, STu),0\}+F(0)$,since $\alpha \in [0,1)$ and $F(0)=0$, we get

$d(u, STu) \leq \alpha\, d(u, STu) < d(u, STu)$,a contradiction ,giving thereby $STu=u$.

Now

$$d(u,(ST)^n x) = d(STu,(ST)^n x)$$

$$\leq d(STu,(ST)^{n+1}x) + d((ST)^{n+1}x,(ST)^n x).$$

Since the pair (S,T) is c.a.r. using (3.4.3.1), STu=u and letting $n \to \infty$, we obtain

$$d(u,(ST)^n x) \leq \alpha \max\{d(u,(ST)^n x),0,0,d(u,(ST)^n x),d(u,(ST)^n x)\}+F(0).$$

$$d(u,(ST)^n x) \leq \alpha\, d(u,(ST)^n x)+F(0).$$

Since $\alpha<1$ and $F(0)=0$, we get $d(u,(ST)^n x) \to 0$ as $n \to \infty$.

Consequently $\{(ST)^n x\}$ converges to u. the remaining part follows from Theorem 3.4.3.

Theorem 3.4.5.

Let S and T be self – mapping of a metric space (X ,d) such that ST is continuous .Then the following conclusions holds:

(a) If a sequence $\{x_n\}$ in X converges to a fixed point z of ST, then $\{x_n\}$ is asymptotically ST – regular.

(b) if $\{x_n\}$ be a sequence in X admitting a subsequence $\{x \quad\}$ with $\lim_i$ $x \quad = z$ and $\lim_i d(STx \quad , x \quad) = 0$, then z is a fixed point of ST. If the pair (S,T) commutes at z and Tz, then Sz and Tz also remains the fixed point of ST.

Proof.

The proof is straightforward, hence it is omitted.

CHAPTER 4

SOME FIXED POINT THEOREMS IN METRICALLY CONVEX SPACES

CHAPTER IV
SOME FIXED POINT THEOREMS IN METRICALLY CONVEX SPACES

4.1.INTRODUCTION

There are several fixed point theorems for single –valued mappings of a closed subset of a Banach space . However , in many applications, the mapping under consideration is not a self-mapping of closed sets . Assad [2] gave sufficient conditions for such single-valued mappings to have a fixed point by proving a fixed point theorem for Kannan's mappings on a Banach space and putting certain boundary conditions on the mapping.

In this chapter we have presented some results from Khan et.al [23,25] which extend the result of Assad [2] for more general single – valued mappings which are also substantial generalization of the main theorem of Chatterjea [10].

4.2. RESULTS FOR METRICALLY CONVEX SPACES.

Definition 4.2.1.

A metric space (X,d) is said to **be metrically convex** if for any $x,y \in X$, with $x \neq y$, there exists $z \in X$, $x \neq z$, $z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y) .$$

The following result is borrowed from Assad and Kirk [4] .In the sequel , stands for the boundary of K .

Lemma 4.2.2 .

If K is a nonempty closed subset of a complete and metrically convex metric space (X,d) , then for any $x \in K$ and $y \notin$,there exists a $z \in$ (the boundary of K) such that $d(x,z)+d(z,y) = d(x,y)$.

Theorem 4.2.3.

Let X be a complete metrically convex space and K a closed non-empty subset of X. Let $T: K \longrightarrow X$ be the mapping satisfying the inequality:

$$d(Tx,Ty) \le C \max \{ d(x,Tx),d(y,Ty))\}+ C'\{d (x,Ty) + d(y,Tx)\} \ . \ . \ . \ (1)$$

for every x,y in K , where C and C' are nonnegative reals such that

$$\max \quad \text{——} , \text{——} \quad = h > 0 ,$$

$$\max \quad \text{——} h, \text{——} h \quad = h' ,$$

$$\max \{ h,h'\} = h'' < 1.$$

Further, for every x in , $Tx \in K$. Then T has a unique fixed point in K.

Proof .

Let $x_o \in K$. Let us construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following manner.

Define $x'_1 = Tx_o$. if $x'_1 \in K$, put $x_1 = x'_1$ if $x'_1 \notin K$,choose $x_1 \in$, so that $d(x_o,x_1) + d(x_1,x'_1) = d(x_o,x'_1)$. Define $x'_2 = Tx_1$. If $x'_2 \in K$, put $x_2 = x'_2$ .If $x'_2 \notin K$, choose $x_2 \in$, so that $d(x_1,x_2)+d(x_2,x'_2)=d(x_1,x'_2)$.Continuing in this way, we obtain $\{x_n\}$ and $\{x'_n\}$ satisfying :

(i) $x'_{n+1}=Tx_n$,

(ii) $x_n=x'_n$, if $x'_n \in K$,

(iii) If $x'_n \notin K$, choose $x_n \in$,so that

$$d(x_{n-1},x_n) + d(x_n, x'_n)= d(x_{n-1},x'_n).$$

put $P = \{x_i \in \{x_n\} : x_i = x'_i\}$,$Q = \{x_i \in \{x_n\} : x_i \neq x'_i\}$,it is not hard to show that if $x_n \in Q$ then x_{n-1} and x_{n+1} belong to P.

Now we wish to estimate $d(x_n,x_{n+1})$.We divide the proof into three cases

Case I.

$x_n,x_{n+1} \in p$. From(1) , we have

$$d(x_n ,x_{n+1}) = d(Tx_{n-1},Tx_n)$$

$$\leq C \max [d(x_{n-1},Tx_{n-1}),d(x_n ,Tx_n)]$$

$$+C'[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]$$

$$= C \max [d(x_{n-1},x_n),d(x_n ,x_{n+1})]$$

$$+C^{'}[d(x_{n-1},x_{n+1})+ d(x_n,x_n)].$$

Then it follows that

$$d(x_n, x_{n+1})\leq C \max [d(x_{n-1},x_n), d(x_n, x_{n+1})] +C'd (x_{n-1}, x_{n+1})$$

$$\leq C \max[d(x_{n-1},x_n), d(x_n, x_{n+1})] +C^{'}[d(x_{n-1},x_n)+d(x_n,x_{n+1})].$$

Now if $d(x_n,x_{n+1}) \leq d(x_{n-1},x_n)$, we have

$$d(x_n,x_{n+1}) \leq C\, d(x_{n-1},x_n)+C^{'}\, d(x_{n-1},x_n) +C^{'}\, d(x_n,x_{n+1}).$$

So $\quad d(x_n,x_{n+1}) \leq [C+C^{'})/1-C^{'})]\, d(x_{n-1},x_n).$

If $\quad d(x_{n-1},x_n) \leq d(x_n,x_{n+1})$,we obtain

$$d(x_n,x_{n+1}) \leq C\, d (x_n,x_{n+1})+C^{'}\, d(x_{n-1},x_n)+C^{'}\, d(x_n,x_{n+1}).$$

So, $\quad d(x_n,x_{n+1}) \leq [C^{'}/1-C-C^{'})]\, d(x_{n-1},x_n).$

Thus in both the situations, we obtain

$d (x_n,x_{n+1}) \leq h\, d (x_{n-1},x_n).$

Case II.

$x_n\in P$, $x_{n+1}\in Q$.

Then condition (1) implies that

$$d (x_n,x_{n+1}) \leq d(x_n,x_{n+1})+d(x_{n+1},x^{'}_{n+1})$$

$= d(x_n,x'_{n+1}) = d(Tx_{n-1},Tx_n)$

$\leq C \max [d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)]$

$+C'[d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})]$

$= C \max [d(x_{n-1},x_n), d(x_n,x'_{n+1})]+C' d(x_{n-1},x'_{n+1}).$

For $d(x_{n-1},x_n) \leq d(x_n,x'_{n+1})$, we have

$d(x_n,x_{n+1}) \leq [C'/1-C-C'] d(x_{n-1},x_n).$

If $d(x_n,x'_{n+1}) \leq d(x_{n-1},x_n)$, we have

$d(x_n,x_{n+1}) \leq [C+C')/(1-C')] d(x_{n-1},x_n).$

Consequently , we have

$d(x_n,x_{n+1}) \leq h\, d(x_{n-1},x_n).$

Case III

$x_n \in Q$, $x_{n+1} \in P$. Then

$d(x_n,x_{n+1}) \leq d(x_n,x'_n) + d(x'_n,x_{n+1})$

$\leq d(x_{n-1},x_n) + d(x_n,x'_n) + d(x'_n,x_{n+1})$

$= d(x_{n-1},x'_n) + d(x'_n,x_{n+1})$

$\leq h\, d(x_{n-2},x_{n-1})+d(x'_n,x_{n+1})$ by case II,since $x_n \in Q$ implies $x_{n-1} \in P$.

Now , if $d(x_{n-1},x'_n) \geq d(x_n,x_{n+1})$, we have

$$d(x_n,x_{n+1}) \leq h\, d(x_{n-2}, x_{n-1}) + d(Tx_{n-1}, Tx_n)$$

$$\leq h\, d(x_{n-2}, x_{n-1}) + C \max[\, d(x_{n-1},x'_n), d(x_n,Tx_n)]$$

$$+C'[d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})]$$

$$= h\, d(x_{n-2},x_{n-1}) + C \max[d(x_{n-1},x'_n), d(x_n,x'_{n+1})]$$

$$+C'[d(x_{n-1}, x'_{n+1}) + d(x_n, x'_n)]$$

$$= h\, d(x_{n-2}, x_{n-1}) + C \max[d(x_{n-1},x'_n), d(x_n,x_{n+1})]$$

$$+C'[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)].$$

Note that here

$$d(x_{n-1}, x_{n+1}) + d(x_n, x'_n) \leq d(x_{n-1},x_n) + d(x_n,x'_n) + d(x_n,x_{n+1}).$$

Hence we obtain

$$d(x_n,x_{n+1}) \leq h\, d(x_{n-2},x_{n-1}) + C\, d(x_{n-1},x'_n) + C'[\, d(x_{n-1},x'_n) + d(x_n,x_{n+1})]$$

which implies

$$d(x_n,x_{n+1}) \leq \text{——}\ d(x_{n-2},x_{n-1}) + \text{——}\ d(x_{n-1},x'_n)$$

$$\leq \text{——}\ d(x_{n-2},x_{n-1}) + \text{——}\, h\ \ d(x_{n-2},x_{n-1})$$

$$\leq \frac{}{} h \quad d(x_{n-2},x_{n-1}).$$

When $d(x_{n-1},x'_n) \leq d(x_n,x_{n+1})$, we see that

$$d(x_n,x_{n+1}) \leq h\, d(x_{n-2},x_{n-1}) + C\, d(x_n,x_{n+1}) + C'\, d(x_{n-1},x'_n) + C'\, d(x_n,x_{n+1}).$$

$$d(x_n,x_{n+1}) \leq \frac{}{} \quad d(x_{n-2},x_{n-1}) + \frac{}{} \quad d(x_{n-1},x'_n)$$

$$\leq \frac{}{} \quad d(x_{n-2},x_{n-1}) + \frac{}{} h \quad d(x_{n-2},x_{n-1})$$

$$\leq \frac{}{} h \quad d(x_{n-2},x_{n-1})$$

Now , combining the above two inequalities, we see that

$$d(x_n,x_{n+1}) \leq h'\, d(x_{n-2},x_{n-1}).$$

Therefore in all three cases we find that

$$d(x_n,x_{n+1}) \leq h''\, \max\{d(x_{n-2},x_{n-1}),d(x_n,x_{n-1})\}.$$

It is routine to verify that for $n > 1$ we obtain

$$d(x_n,x_{n+1}) \leq (h'')^{n/2} \quad , \text{where} \quad = (h'')^{-1/2} \max[\, d(x_o,x_1),d(x_1,x_2)].$$

Thus for $m > n > N$,

$$d(x_m,x_n) \leq \sum \quad d(x \;, x \quad) \leq \sum \quad (h \;)^{/} \quad \delta .$$

Therefore $\{x_n\}$ is a Cauchy sequence , and hence converges to a limit p.

Also there exists an infinite subsequence $\{x_{n(k)}\}$of $\{x_n\}$ such that $x_{n(k)+1} \in P$.

Then

$$d(Tp,p) \leq d(Tp,Tx_{n(k)}) + d(x_{n(k)+1},p)$$

$$\leq C \max\{d(p,Tp), d(x_{n(k)},Tx_{n(k)})\}$$

$$+C'\{d(p,Tx_{n(k)})+d(x_{n(k)},Tp)\}+d(x_{n(k)+1},p).$$

Letting k tends to infinity , we get

$$d(Tp,p) \leq (C+C') d(Tp,p) +C' d(Tp,p),$$

which implies that $Tp=p$, since $h<1$.

Condition (1) ensures that p is also unique.

This completes the proof.

Theorem 4.2.4.

Let X be a complete metrically convex space and K a closed non -empty subset of X. Let $T:K \longrightarrow X$ be a mapping satisfying the inequality:

$$d(Tx,Ty) \leq C \max[1/2d(x,y),d(x,Tx),d(y,Ty)+C'[d(x,Ty)+d(y,Tx)] \ldots\ldots(2)$$

For every x,y in K , where C and C' are non - negative reals with $\max[(C+C')/(1-C'), C'/(1-C-C')] = h < 1$. Further ,for every x in ,Tx∈K. Then T has a unique fixed point in K

proof.

Let $x_o \in K$. Let us construct the two sequences $\{x_n\}$ and $\{x'_n\}$ in the following manner:

Define $x'_1 = Tx_o$. if $x'_1 \in K$, put $x_1 = x'_1$ if $x'_1 \notin K$, choose $x_1 \in$, so that $d(x_o,x_1) + d(x_1,x'_1) = d(x_o,x'_1)$. Define $x'_2 = Tx_1$. If $x'_2 \in K$, put $x_2 = x'_2$. If $x'_2 \notin K$, choose $x_2 \in$, so that $d(x_1,x_2)+d(x_2,x'_2) = d(x_1,x'_2)$. Continuing in this way ,we obtain $\{x_n\}$ and $\{x'_n\}$ satisfying :

(i) $x'_{n+1} = Tx_n$,

(ii) $x_n = x'_n$, if $x'_n \in K$,

(iii) If $x'_n \notin K$, choose $x_n \in$,so that

$$d(x_{n-1},x_n) + d(x_n, x'_n) = d(x_{n-1},x'_n).$$

put $P = \{x_i \in \{x_n\} : x_i = x'_i\}$, $Q = \{x_i \in \{x_n\} : x_i \neq x'_i\}$ it is not hard to show that if $x_n \in Q$ then x_{n-1} and x_{n+1} belong to P. Now we wish to estimate $d(x_n,x_{n+1})$.

Case I.

$x_n, x_{n+1} \in P$. From(2) , we have

$$d(x_n, x_{n+1}) == d(Tx_{n-1}, Tx_n)$$

$$\leq C \max [1/2 d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)]$$

$$+C'[d(x_{n-1} Tx_n) + d(x_n, Tx_{n-1})]$$

$$= C \max [d(x_{n-1}, x_n), d(x_{n,} x'_{n+1})]$$

$$+C'[d(x_{n-1}, x_{n+1}) + d(x_n, x'_n)].$$

Then it follows that

$$d(x_n, x_{n+1}) \leq C \max [d(x_{n-1}, x_n), d(x_n, x_{n+1})] + C' d(x_{n-1}, x_{n+1})$$

$$\leq C \max[d(x_{n-1}, x_n), d(x_n, x_{n+1})] + C'[d(x_{n-1}, x_n) + d(x_n, x_{n+1})].$$

Now if $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, we have

$$d(x_n, x_{n+1}) \leq C\, d(x_{n-1}, x_n) + C'\, d(x_{n-1}, x_n) + C'\, d(x_n, x_{n+1}).$$

So $d(x_n, x_{n+1}) \leq [C+C')/1-C')]\, d(x_{n-1}, x_n)$.

When $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$,we obtain

$$d(x_n, x_{n+1}) \leq C\, d(x_n, x_{n+1}) + C'\, d(x_{n-1}, x_n) + C'\, d(x_n, x_{n+1}).$$

So $\quad d(x_n,x_{n+1}) \leq [C'/1-C-C'] \, d(x_{n-1},x_n)$.

Thus in both the situations, we obtain

$$d\,(x_n,x_{n+1}) \leq h\, d\,(x_{n-1},x_n).$$

Case II.

$x_n \in P$, $x_{n+1} \in Q$.

Then condition (2) implies that

$$d\,(x_n,x_{n+1}) \leq d(x_n,x_{n+1})+d(x_{n+1},x'_{n+1})$$

$$=d(x_n,x'_{n+1})= d\,(Tx_{n-1},Tx_n)$$

$$\leq C \max [1/2 \;\; d\,(x_{n-1},x_n),d\,(x_{n-1},Tx_{n-1}),d\,(x_n,Tx_n)]$$

$$+C'[d\,(x_{n-1},Tx_n)+d\,(x_n,Tx_{n-1})]$$

$$= C \max [\, d\,(x_{n-1},x_n),d\,(x_n,x'_{n+1})]+C'\, d\,(x_{n-1},x'_{n+1}).$$

For $\quad d\,(x_{n-1},x_n) \leq d\,(x_n,x'_{n+1})$, we have

$$d\,(x_n,x_{n+1}) \leq [C'/1-C-C'] \, d\,(x_{n-1},x_n).$$

If $\quad d\,(x_n,x'_{n+1}) \leq d\,(x_{n-1},x_n)$, we have

$$d\,(x_n,x_{n+1}) \leq [C+C')\,/(1-C')] \, d\,(x_{n-1},x_n).$$

Consequently , we get

$$d(x_n,x_{n+1}) \le h\, d(x_{n-1},x_n).$$

Case III.

$x_n \in Q$, $x_{n+1} \in P$.

As $x_n \in Q$ is a convex linear combination of x_{n-1} and x'_n , we have

$$d(x_n ,x_{n+1}) \le \max\ [d(x_{n-1},x_{n+1}), d(x'_n,x_{n+1})].$$

If $d(x_{n-1},x_{n+1}) \le d(x'_n,x_{n+1})$, we have

$$d(x_n,x_{n+1}) \le d(x'_n,x_{n+1})$$

$$= d(Tx_{n-1}, Tx_n).$$

$$\le C \max [1/2\, d(x_{n-1},x_n), d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n)]$$

$$+ C' [d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})]$$

$$= C \max [1/2\, d(x_{n-1},x_n), d(x_{n-1},x'_n), d(x_n,x'_{n+1})]$$

$$+ C' [d(x_{n-1},x'_{n+1}) + d(x_n,x'_n)]$$

$$= C \max [d(x_{n-1}, x'_n), d(x_n ,x_{n+1})]$$

$$+ C'[d(x_{n-1} ,x_{n+1}) + d(x_n , x'_n)].$$

Note that here

$$d(x_{n-1}, x_{n+1}) + d(x_n , x'_n) \le d(x_{n-1}, x_n) + d(x_n,x_{n+1}) + d(x_n,x'_n)$$

$$\le d(x_{n-1},x'_n) + d(x_n,x_{n+1}).$$

If $d(x_n,x_{n+1}) \leq d(x_{n-1},x'_n)$, we have

$$d(x_n,x_{n+1}) \leq C\, d(x_{n-1},x'_n)+C'[d(x_{n-1},x'_n)+d(x_n,x_{n+1})].$$

Whence $d(x_n,x_{n+1}) \leq [C+C']/(1-C')]\, d(x_{n-1},x'_n)$.

When $d(x_{n-1},x'_n) \leq d(x_n,x_{n+1})$, we have

$$d(x_n,x_{n+1}) \leq C\, d(x_n,x_{n+1})+C'\, d(x_{n-1},x'_n)+C'\, d(x_n,x_{n+1}).$$

That is $d(x_n,x_{n+1}) \leq [C'/(1-C-C')]d(x_{n-1},x'_n)$.

Therefore ,we have

$$d(x_n,x_{n+1}) \leq h\, d(x_{n-1},x'_n)$$

$d(x_n,x_{n+1}) \leq h^2 d(x_{n-2},x_{n-1})$ by case II, since $x_n \in Q$ implies $x_{n-1} \in P$.

Now if $d(x'_n,x_{n+1}) \leq d(x_{n-1},x_{n+1})$ we have

$$d(x_n,x_{n+1}) \leq d(x_{n-1},x_{n+1})$$

$$= d(Tx_{n-2},Tx_n)$$

$$\leq C \max[1/2d(x_{n-2},x_n),d(x_{n-2},Tx_{n-2}),d(x_n,Tx_n)]$$

$$+C'[d(x_{n-2},Tx_n)+d(x_n,Tx_{n-2})].$$

$$=C \max[1/2d(x_{n-2},x_n),d(x_{n-2},x'_{n-1}),d(x_n,x'_{n+1})]$$

$$+C'[d(x_{n-2},x'_{n+1})+d(x_n,x'_{n-1})].$$

Clearly, we have

$$1/2\ d\ (x_{n-2},x_n) \leq 1/2[\ d\ (x_{n-2},x_{n-1})+d\ (x_{n-1},x_n)]$$

$$\leq \max\ [d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)].$$

Therefore one gets

$$d(x_n,x_{n+1}) \leq C \max[d(x_{n-2},x_{n-1})\ d(x_{n-1},x_n),d(x_{n-2},x_{n-1}),\ d(x_n,x_{n+1})]$$

$$+\ C'[d(x_{n-2},x_{n+1})+d(x_n,x_{n-1})].$$

$$\leq\ C \max\ [d(x_{n-2},x_{n-1})+d(x_n,x_{n+1})]$$

$$+C'[d(x_{n-2},x_{n+1})+d(x_n,x_{n-1})].$$

If $d(x_n,x_{n+1}) \leq d(x_{n-2},x_{n-1})$, we have

$$d(x_n,x_{n+1}) \leq\ C\ d(x_{n-2},x_{n-1})+C'[d(x_{n-2},x_{n-1})+d(x_{n-1},x_{n+1})+d(x_n,x_{n-1})]$$

$$\leq\ C\ d(x_{n-2},x_{n-1})+C'd(x_{n-2},x_{n-1})+C'd(x_n,x_{n+1}).$$

Thus $d(x_n,x_{n+1}) \leq [(C+C')/(1-C')]\ d(x_{n-2},x_{n-1})$.

When $d(x_{n-2},x_{n-1}) \leq d(x_n,x_{n+1})$, we obtain

$$d(x_n,x_{n+1}) \leq C\ d(x_n,x_{n+1})+C'\ d(x_{n-2},x_{n-1})+C'\ d(x_n,x_{n+1}).$$

Which yields

$d(x_n,x_{n+1}) \leq [C'/(1-C- C')]d(x_{n-2},x_{n-1})$.

Now combining the above two inequalities , we see that

$d(x_n,x_{n+1}) \leq h\, d(x_{n-2},x_{n-1})$.

Therefore in all three cases ,we find that

$d(x_n,x_{n+1}) \leq h \max [d(x_{n-2},x_{n-1}), d(x_n,x_{n-1})]$.

It is routine to verify that for $n >1$,

$d(x_n,x_{n+1}) \leq h^{n/2}$, where $=h^{-1/2} \max [d(x_o,x_1),d(x_1,x_2)]$.

Thus for $m >n >N$,

$d(x_m,x_n) \leq \sum \quad d(x\ ,x\quad) \leq \sum \quad h^{/}$.

Therefore $\{x_n\}$ is a Cauchy sequence,and hence converges to a limit p(say).

Also there exists an infinite subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)+1} \in P$.

Then

$d(Tp,p) \leq d(Tp,Tx_{n(k)})+d(x_{n(k)+1},p)$

$\leq C \max[1/2\, d(p,x_{n(k)}),d(p,Tp),d(x_{n(k)},Tx_{n(k)})]$

$+ C' [d(p,Tx_{n(k)})+d(x_{n(k)},Tp)]+d(x_{n(k)+1},p)$.

Letting k tends to infinity , we have Tp=p .Condition (2) ensures that p is also unique .This completes the proof.

REFERENCES

[1] Aksoy, A. And Khamsi, M.A.,Non standard methods in fixed point theory, Springer Verleg, Berlin, 1990.

[2] Assad , N.A. On a fixed point theorem of Kannan in Banach spaces. Tamkang J .Math.7(1976),91-94.

[3] Asaad, N .A .Fixed point theorems for set-valued transformations on compact sets. Boll .Un. Math .Ital .(4)7(1973),1-7.

[4] Assad , N.A . And Kirk ,W.A. fixed point theorems for set - valued mappings of contractive type, Pacific J. Math.,43(3) (1972), 553-562.

[5] Banach ,S .Sur Les operations dans les ensembles abstraits et Leur application aux equations integrals, Funda Math., 3(1922) ,133-181.

[6] Bohnen blust, H.F.And Karlin ,S.,on a theorem of Ville Contributions to the theory of games,(Editor :Kuhn and Tucker, Univ. Press, Princeton), I(1950) , 155-160.

[7] Brouwer, L.E.J . Under Abbildung Von Mannigfalting Keiten , Math. Ann.,71 (1912), 97-115.

[8] Browder, F.E.And Petryshyn, M.V.,The solution by iteration of nonlinear functional equations in Banach space , Bull. Amer. Math.Soc., 72 (1966), 571-575.

[9] Brown ,A.L .And A.Page., Elements of functional analysis ,Van Nostrand Reinhold company London(1970) .

[10] Chatterjea, S.K.Fixed point theorems ,C.R. Acad. Bulgare Sci 25(1972), 727-730.

[11] Ciric ,Lj.B. Ageneralization of Banach Ts Contraction Principle. Math. Soc. 45(1974),267-273.

[12] Dugundji, J .And Granas ,A., Fixed point theory I, Polish Scientific Publisher Warsawa (1982).

[13] Edelstein, M.on fixed and periodic points under contraction mappings,

Jour . London Math . Soc ., 37 (1962) , 74-79.

[14] Engl, H.W.,Weak convergence of asymptotically regular sequences for non-expansive mappings and connections with certain Chebishef centres, Nonlinear Analysis, Theory, Methods and Applications, Vol. 1, No. 5 (1977), 495-501.

[15] Fan , Ky., Fixed point and minimax theorems in locally convex topological linear spaces,Proc. Nat. Acad. Sci.U.S.A., 38 (1952), 121-126.

[16] Glicksberg, I.,A further generalization of Kakutani fixed point theorem with applications to Nash equilibrium points, Proc. Amer .Math. Soc., 3 (1952), 170-174.

[17] Goebel , K .And Kirk ,W .A .,Topics in metric fixed point theory, Cambridge University Prees, London (1990).

[18] Górnicki ,j. (Rzeszów), A fixed point theorem for asymptotically regular mappings, Colloquium Mathematican ,Vol. LXIV(1993), FASC.1, 55-57.

[19] Hardy,G.E. And Rogers, T.D. ;A generalization of a fixed point theorem of Reich, Canad. Math. Bull., 16 (1973), 201-206.

[20] Hillam ,O.P. A characterization of the converges on successive approximations, Amer. Math .Monthly ,83(1976) , 273-276.

[21] Istratescu, V.I., Fixed point theory, An introduction , D. Reidel Publ. Comp. (1981).

[22] Imdad ,M. Khan ,T,I., Composite asymptotic regularity and common fixed points ,Italian journal of pure and applied mathematics-N.11-2002 (159-164).

[23] Khan, M.S. Pathak .H.K. And Khan ,M .D;Some fixed point theorems in metrically convex spaces , Georgian Math.J.,7(3)(2000),523-530.

[24] Khan ,M.S .Khan .M.D., Fixed point theorems ,Communications de la Faculte' des science ,Ankara, Vol.33 (1983),115-120

[25] Khan ,M .D. And Bharadwaj , R.,A fixed point theorem in metrically

convex space Indian J.Math.,43(3)(2001),373-379.

[26] Khan ,M.D.And Bharadwaj, R.,Some common fixed point theorems ,Journal, Ganita ,Vol. 51 No 2(2000),143-148.

[27] Kannan ,R. Some results on fixed points I ,Bull Cal. Math. Soc .,60 (1968), 71-76.

[28] Kannan , R. Some results on fixed points II ,Amer. Math.Monthly, 76 (1969) , 405-408.

[29] Kolmogrov and S.J. Fomin, A.M. Functional Analysis Vol.1, Graylock Rochester, (1957).

[30] Kakutani, S., A generalization of Brouwer's fixed point theorem , Duke Math. J., 8 (1941), 457-459.

[31] Kinoshita ,S .On some contractible continua without the fixed point property, Funda. Math., 40 (1953), 96-98.

[32] Park,S.,Eighty years of the Brouwer fixed point theorem , Lecture Notes Series (28) (Antipodal Points and Fixed points), Research

lnstitute of Mathematics Global, Analysis Research Center , Seoul National University,seoul(Korea).

[33] Rhoades, B.E.A comparison of various definitions of contractive mappings ,Trans .Amer .Math .Soc., 26 (1977) , 256-290.

[34] Rhoades, B.E. Contractive definitions revisited, contemporary Mathematics, 21 (1983) , 189-205.

[35] Reich, S. Come results concerning contraction mappings , Canad . Math . Bull., 14 (1971) , 121-124.

[36] Reich, S., Kannan's fixed point theorem , Boll. Un . Math. Ital . 4(1971) , 1-11.

[37] Robert , G.Bartle .,Elements of Real Analysis , John Wiley and Sons , Inc New York . London . Sydney (1967) .

[38] Rus, I.A.,Some fixed point theorems in metric spaces, Rend. Ist. Mat.Uni. Trieste, 3 (1971), 169-172.

[39] Schauder , J.,Der fixpunktsatz functional raumen, Studia Math., 2 (1930), 171-180.

[40] Smart , D.R., Fixed point theorems, Cambridge University Press, 1974.

[41] Tychonoff, A., Ein fixpunktsatz, Math. Ann., 111(1935), 767-776.

الخلاصه

فى هذه الأطروحه ناقشنا مفهوم نظريات النقطه الثابته والدوال المتعلقه بها ,حيث أن في الفصل الأول عرضنا بعض المفاهيم والنظريات الأساسيه التى استعملت خلال الأطروحه .

أماالفصل الثاني عرضنا بعض نظريات النقطه الثابته لدوال القيمه المفرده والتي هى تعميم لنظريتي Banach[5] وKannan[27] .

وفي الفصل الثالث ناقشنا مفهومي الشكل المقارب للمتسلسلات المنتظمه والشكل المقارب للدوال المنتظمه والتي هي تعميم لنظريه Hardy-Roger's [19] .

وفي الفصل الرابع دمجنا نظريات النقطه الثابته إلى الفراغات المحدبه المتريه والتي هى تعميم لنظريه Assad [(1),(2),(3)] وChatterjia [10] وKannan [(27),(28)].

وفي النهايه عرضنا بعض المراجع التي استعملت خلال هذه الاطروحه .

والله ولي التوفيق

جامعة بنغازي

كلية العلوم

قسم الرياضيات

Asymptotically Regular Mappings And The Fixed Point Theorems

مقدم للاستيفاء الجزئي لمتطلبات درجة التخصص العالي (الماجستير) في الرياضيات

مقدم من

ايمان ابراهيم الهمالي

اشراف

د.محمد الدين خان

بنغازي - ليبيا

2012-2011