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**The (G'/G) Expansion Method for
Solving Nonlinear Partial Differential Equations**

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Dedication

This thesis is dedicated to my beloved parents,

my husband, my child

my sisters and my brother.

my colleagues,

and to all who helped me to finish my work.

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Abstract

The (G'/G) -expansion method, for solving nonlinear evolution equations is reviewed and applied to solve selected equations, which have positive integer homogeneous balance number. For the equations with non integer homogeneous balance number, we suggested an approach to write a transform to reduce the homogeneous balance number to a positive integer, and as an application, we apply this approach successfully to find a traveling wave solution to the general KdV equation for $p = 3$. For both cases and with help of Maple tools, we compared our results with the results obtained by other methods. Theses study show that this method is an effective method, and can be used to solve many types of nonlinear evolution equations.

Introduction

The solution of nonlinear evolution equations (NLEEs), which is an equation on a continuous time variable t , are significance in mathematical physics and becomes one of the most exciting and extremely active areas of research investigation. In the several decades ago, many useful methods for obtaining exact solutions of (NLEEs) have been presented, such as the inverse scattering method (**Ablowitz & Clarkson,1991**), the Hirota's bilinear transform method (**R. Hirota,1971 & Yan,2011**), the Bäcklund transformation method(**Mimura, 1978, Rogers & Shadwick, 1982**), the truncated Painlevé expansion method(**Weiss, Tabor & Carnevale,1982, Zhang, Wu & Lou,2002**), the sine-cosine method (**Wazwaz,2005**), the homogeneous balance method(**Wang, Zhou& Li,1996**), the tanh-coht method (**Malfliet,1992, Abdel-All, Abdel Razek &Seddeek,2011**), the direct algebraic method(**Soliman & Abdo,2009**), the Jacobi elliptic function expansion method(**Liu, Fu, Liu & Zhao,2001, Yan, 2003**), the F-expansion method (**Wang & Zhou, 2003, Wang & Li, 2005**), and the exp-function expansion method (**He & Wu,2006**). However, most of these methods used to solve the (NLEEs)with constant coefficients and in recent years, the (NLEEs)with variable coefficients are studied.

Recently, proposed the (G'/G) -expansion method to find travelling wave solutions of NLEEs. These method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in (G'/G) , where the function $G = G(\zeta)$ satisfies a second order linear ordinary differential equation, by which the travelling wave solutions involving parameters. The travelling wave for special values of these parameters gives the solitary wave solution. The travelling wave solution is expressed in term of hyperbolic functions, trigonometric functions and rational functions(**Hunter,1996**).

In chapter one, we reivewedthe (G'/G) -expansion method and apply the method on some selected (NLEEs).

In chapter Two, we we,introduced the general form of the KdV equation and solve the equation for $p = 1$.

In chapter three,we applied the (G'/G) -expansion method on Boussinesq's equation and variant Boussinesq's equation.

In chapter four, we applied the (G'/G) -expansion method to thE general KdV equation for $p = 3$.

Chapter One

The (G'/G) Expansion Method

In this chapter, we review the (G'/G) -expansion method for solving the nonlinear evolution equations (Hunter, 1996). And, apply the method to find travelling wave solutions for selected nonlinear evolution equations.

(1.1) The Method

To introduce the (G'/G) -expansion method (Wang, Li & Zhang, 2008), we first define the nonlinear partial differential equation

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (1.1)$$

In (1.1), P is a polynomial in $u(x, t)$ and its partial derivatives, in which it involves highest order derivatives and nonlinear terms. To solve (1.1) by using the (G'/G) -expansion method, we follow the steps:

Step (1):

First, we introduce the traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x - Vt, \quad (1.2)$$

And then using (1.2), with the chain rule, we reduce (1.1) to the following ordinary differential equation

$$F(u(\xi), -Vu'(\xi), u'(\xi), V^2u''(\xi), -Vu''(\xi), u''(\xi), \dots) = 0, \quad u^{(i)}(\xi) = \frac{d^i u(\xi)}{d\xi^i}, \quad (1.3)$$

where V is the speed of the traveling wave and F is a polynomial in $u(\xi)$ and its derivatives.

Step (2):

Then, we suppose that the solution of the ordinary differential equation (1.3) can be expressed by a polynomial in (G'/G) of degree m of the form:

$$u(\zeta) = \sum_{i=0}^m \alpha_i (G'/G)^i . \quad (1.4)$$

In (1.4), m is a positive integer called the homogeneous balance number, which is determined by considering the homogeneous balance between the highest order derivatives and the highest nonlinear terms appearing in (1.3), and the function $G = G(\zeta)$ satisfies a second order linear ordinary differential equation of the form:

$$G'' + \lambda G' + \mu G = 0 , \quad (1.5)$$

where λ , μ and α_i ; ($i=0,1,2,\dots,m$), are constants to be determined provided that $\alpha_m \neq 0$.

Note that, in the case, when the homogeneous balance number m is not a positive integer (**Zayed & Alurfi, 2014**), we use transformation

$$u(\xi) = \Phi^m(\xi) \quad (1.6)$$

For example the general KdV equation, $u_t + (p+1)(p+2)u^p u_x + u_{xxx} = 0$, $p = 3$, has homogeneous balance number $m = 2/3$, in this case, we follow (**Drazin & Johnson, 1983**), and used the transform (1.6).

Step(3):

Next, we substitute (1.4) into (1.3) and make use of the second order equation (1.5), then collect all the coefficients of (G'/G) , which reduce (1.3) to a polynomial in

(G'/G) . Then the resulting is a system of algebraic equations of V, λ, μ and $\alpha_i; (i=0,1,2,\dots,m)$.

Step (4):

We calculate the constants V, λ, μ and $\alpha_i; (i=0,1,2,\dots,m)$ and substitute of these values into (1.5) and solve it. At the end, by using (1.4), we get exact solutions of the nonlinear evolution equation (1.1).

(1.2) Numerical Examples

In this section, we apply (G'/G) -expansion method to selected non-linear evolution equations.

Example (1.1): Consider the Burger's equation (**Drazin & Johnson, 1983**)

$$u_t + uu_x - u_{xx} = 0 \tag{1.7}$$

Using a transformation (1.2), then (1.7) reduce to the ordinary differential equation

$$-Vu' + uu' - u'' = 0, \tag{1.8}$$

which by integrating, will reduce to

$$C - Vu + \frac{1}{2}u^2 - u' = 0, \tag{1.9}$$

Hence the homogeneous balance between u^2 and u' is $(2m = m + 1 \Rightarrow m = 1)$. To find the general solution of the equation (1.8), we use (1.5). Now for $m=1$, (1.4) reads

$$u(\xi) = \alpha_0 + \alpha_1(G'/G), \tag{1.10}$$

To substitute (1.10) into (1.9), we calculate

$$u' = \alpha_1 (G'/G)' = \alpha_1 \left(\frac{GG'' - G'G'}{G^2} \right) = \alpha_1 \left((G''/G) - (G'/G)^2 \right) \quad (1.11)$$

Then, by using (1.4), we can write

$$G'' = -\lambda G' - \mu G \quad (1.12)$$

This reduce (1.11) to

$$u' = -\alpha_1 (G'/G)^2 - \alpha_1 \lambda (G'/G) - \alpha_1 \mu \quad (1.13)$$

Then, by using (1.10), we can find u^2 as following

$$u^2 = \left[\alpha_1 (G'/G) + \alpha_0 \right]^2 = \alpha_1^2 (G'/G)^2 + 2\alpha_1 \alpha_0 (G'/G) + \alpha_0^2 \quad (1.14)$$

By substituting (1.10), (1.13) and (1.14) into (1.9) we obtain

$$C - V \left(\alpha_1 (G'/G) + \alpha_0 \right) + \frac{1}{2} \left(\alpha_1^2 (G'/G)^2 + 2\alpha_1 \alpha_0 (G'/G) + \alpha_0^2 \right) - \left(-\alpha_1 (G'/G)^2 - \alpha_1 \lambda (G'/G) - \alpha_1 \mu \right) = 0,$$

which can be simplify to

$$\left(\alpha_1 + \frac{1}{2} \alpha_1^2 \right) (G'/G)^2 + (\alpha_1 \alpha_0 - V \alpha_1 + \alpha_1 \lambda) (G'/G) + C - V \alpha_0 + \frac{1}{2} \alpha_0^2 + \alpha_1 \mu = 0 \quad (1.15)$$

Equating each coefficients of (1.15) to zero, yields the system of algebraic equations:

$$\begin{aligned}
C - V\alpha_0 + \frac{1}{2}\alpha_0^2 + \alpha_1\mu &= 0 \\
\alpha_1\alpha_0 - V\alpha_1 + \alpha_1\lambda &= 0 \\
\alpha_1 + \frac{1}{2}\alpha_1^2 &= 0
\end{aligned} \tag{1.16}$$

Solving these system of the algebraic equations for α_1 , V and C

$$\begin{aligned}
\alpha_1 + \frac{1}{2}\alpha_1^2 = 0 &\rightarrow 2\alpha_1 + \alpha_1^2 = 0 \rightarrow \alpha_1 = -2, \text{ hence } \alpha_1 \neq 0 \\
\alpha_1\alpha_0 - V\alpha_1 + \alpha_1\lambda = 0 &\rightarrow -2\alpha_0 + 2V - 2\lambda = 0 \rightarrow V = \alpha_0 + \lambda, \\
C - \lambda\alpha_0 - \alpha_0^2 + \frac{1}{2}\alpha_0^2 - 2\mu = 0 &\rightarrow C = \lambda\alpha_0 + \frac{1}{2}\alpha_0^2 + 2\mu, \alpha_0 = \alpha_0
\end{aligned} \tag{1.17}$$

And λ , μ and α_0 are arbitrary constants.

Then (1.10) become

$$u(\xi) = -2(G'/G) + \alpha_0, \quad \xi = x - (\alpha_0 + \lambda)t, \tag{1.18}$$

Now, the auxiliary equation of (1.5) has the roots

$$m_{1,2} = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2} \tag{1.19}$$

For the solution of (1.5), we have the following cases

Case (I) $\lambda^2 - 4\mu = \omega_1^2 > 0$

For this case we have

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right), \quad (1.20)$$

and

$$G' = e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right) \quad (1.21)$$

This gives

$$(G'/G) = \frac{e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right)} \quad (1.22)$$

And, by using

$$\cosh \frac{\omega_1}{2} \xi + \sinh \frac{\omega_1}{2} \xi = e^{\frac{\omega_1}{2}\xi}, \quad \cosh \frac{\omega_1}{2} \xi - \sinh \frac{\omega_1}{2} \xi = e^{-\frac{\omega_1}{2}\xi}, \quad (1.23)$$

we can write

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{c_1 \cosh \frac{\omega_1}{2} \xi + c_1 \sinh \frac{\omega_1}{2} \xi - c_2 \cosh \frac{\omega_1}{2} \xi + c_2 \sinh \frac{\omega_1}{2} \xi}{c_1 \cosh \frac{\omega_1}{2} \xi + c_1 \sinh \frac{\omega_1}{2} \xi + c_2 \cosh \frac{\omega_1}{2} \xi - c_2 \sinh \frac{\omega_1}{2} \xi} \right) \quad (1.24)$$

Next if we set

$$B_1 = c_1 + c_2, \quad B_2 = c_1 - c_2, \quad (1.25)$$

Then, (1.24) can be written in the form

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{B_1 \sinh \frac{\omega_1}{2} \xi + B_2 \cosh \frac{\omega_1}{2} \xi}{B_1 \cosh \frac{\omega_1}{2} \xi + B_2 \sinh \frac{\omega_1}{2} \xi} \right), \quad (1.26)$$

or

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} K_1, \quad (1.27)$$

where

$$K_1 = \left(\frac{B_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + B_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{B_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + B_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right) \quad (1.28)$$

Now the solution (1.18) has the form

$$u(\zeta) = -\omega_1 K_1 + \lambda + \alpha_0, \quad (1.29)$$

or

$$u(\zeta) = -\sqrt{\lambda^2 - 4\mu} K_1 + \lambda + \alpha_0 \quad (1.30)$$

If, we take $B_1 \neq 0$, $B_2 = 0$, then (1.30) reads

$$u(\zeta) = -\sqrt{\lambda^2 - 4\mu} \left(\frac{\sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{\cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right) + \lambda + \alpha_0$$

$$u(\xi) = -\sqrt{\lambda^2 - 4\mu} \left(\frac{e^{\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi} - e^{-\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi}}{e^{\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi} + e^{-\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi}} \right) + \lambda + \alpha_0$$

This leads to

$$\begin{aligned} u(\zeta) &= -\sqrt{\lambda^2 - 4\mu} \left(\frac{e^{\sqrt{\lambda^2 - 4\mu}\zeta} - 1}{e^{\sqrt{\lambda^2 - 4\mu}\zeta} + 1} \right) + \lambda + \alpha_0 \\ &= \left(\frac{-\sqrt{\lambda^2 - 4\mu}e^{\sqrt{\lambda^2 - 4\mu}\zeta} + \sqrt{\lambda^2 - 4\mu}}{e^{\sqrt{\lambda^2 - 4\mu}\zeta} + 1} \right) + \lambda + \alpha_0, \\ &= \frac{-\sqrt{\lambda^2 - 4\mu}(e^{\sqrt{\lambda^2 - 4\mu}\zeta}) + \sqrt{\lambda^2 - 4\mu} + \lambda e^{\sqrt{\lambda^2 - 4\mu}\zeta} + \lambda + \alpha_0 e^{\sqrt{\lambda^2 - 4\mu}\zeta} + \alpha_0}{e^{\sqrt{\lambda^2 - 4\mu}\zeta} + 1} \end{aligned}$$

and the solution is

$$u(\zeta) = \frac{(\lambda + \alpha_0 - \sqrt{\lambda^2 - 4\mu})e^{\sqrt{\lambda^2 - 4\mu}\zeta} + (\sqrt{\lambda^2 - 4\mu} + \lambda + \alpha_0)}{e^{\sqrt{\lambda^2 - 4\mu}\zeta} + 1}, \quad (1.31)$$

Case (II) $\lambda^2 - 4\mu = -\omega_2 < 0 \quad \Rightarrow \omega_2 = 4\mu - \lambda^2.$

For this case we have the solution

$$G = e^{-\frac{\lambda}{2}\zeta} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \zeta + c_2 \sin \frac{\sqrt{\omega_2}}{2} \zeta \right), \quad (1.32)$$

And

$$G' = e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right), \quad (1.33)$$

This gives

$$(G'/G) = \frac{\left(e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right) \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right)} \quad (1.34)$$

If we set (1.25), then we have

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{B_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - B_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{B_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + B_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) \quad (1.35)$$

Then (1.35) can be written as:

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2, \quad (1.36)$$

where

$$K_2 = \begin{pmatrix} B_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - B_1 \sin \frac{\sqrt{\omega_2}}{2} \xi \\ B_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + B_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \end{pmatrix} \quad (1.37)$$

Then the solution (1.18) become

$$u(\zeta) = -\sqrt{\omega_2} K_2 + \lambda + \alpha_0, \quad (1.38)$$

and by using $\omega_2 = 4\mu - \lambda^2$, we can write the solution in the form

$$u(\zeta) = -\sqrt{(4\mu - \lambda^2)} K_2 + \lambda + \alpha_0 \quad (1.39)$$

Case (III) $\lambda^2 - 4\mu = 0$

For this case, we get the solution

$$G = c_1 e^{-\frac{\lambda}{2}\xi} + c_2 \xi e^{-\frac{\lambda}{2}\xi} \quad (1.40)$$

And

$$G' = -\frac{\lambda}{2} c_1 e^{-\frac{\lambda}{2}\xi} + c_2 e^{-\frac{\lambda}{2}\xi} - \frac{\lambda}{2} c_2 \xi e^{-\frac{\lambda}{2}\xi} \quad (1.41)$$

This gives

$$(G'/G) = \frac{-\frac{\lambda}{2} c_1 + c_2 - \frac{\lambda}{2} c_2 \xi}{c_1 + c_2 \xi} = -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \quad (1.42)$$

Then the solution (1.18) reads:

$$u(\zeta) = -\frac{2c_2}{c_1 + c_2\xi} + \lambda + \alpha_0 \quad (1.43)$$

Example(1.2): Consider the celebrated Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation (**Shakeel & Mohyud-Din, 2015**)

$$u_t + u_x - 2auu_x - bu_{xxt} = 0 \quad (1.44)$$

Using a transforme (1.2), then (1.44) reduce to the ordinary differential equation

$$-Vu' + u' - 2auu' - b(-V)u''' = 0 \quad (1.45)$$

By integration (1.45) with respect to ξ , we find

$$C -Vu + u - au^2 + bVu'' = 0 \quad (1.46)$$

Considering the homogeneous balance between u'' and u^2 in (1.46) hence $(2m = m + 2 \Rightarrow m = 2)$. In this case (1.4) reads

$$u(\zeta) = \sum_{i=0}^2 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2, \quad \alpha_2 \neq 0 \quad (1.47)$$

which can be written in the form

$$\begin{aligned} u^2(\zeta) &= \left[\alpha_2 (G'/G)^2 + \alpha_1 (G'/G) + \alpha_0 \right]^2 \\ &= \alpha_2^2 (G'/G)^4 + 2\alpha_1\alpha_2 (G'/G)^3 \\ &+ (\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 + 2\alpha_1\alpha_0 (G'/G) + \alpha_0^2 \end{aligned} \quad (1.48)$$

By using (1.12) and (1.47), we can find

$$u' = -2\alpha_2(G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 - (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu \quad (1.49)$$

$$u'' = 6\alpha_2(G'/G)^4 + (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) + \alpha_1\lambda\mu + 2\alpha_2\mu^2 \quad (1.50)$$

By substituting (1.48), (1.50) and (1.47) into (1.46)

$$\begin{aligned} & C - V \left(\alpha_2(G'/G)^2 + \alpha_1(G'/G) + \alpha_0 \right) \\ & + \left(\alpha_2(G'/G)^2 + \alpha_1(G'/G) + \alpha_0 \right) \\ & - a \left(\alpha_2^2(G'/G)^4 + 2\alpha_1\alpha_2(G'/G)^3 + (\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 + 2\alpha_1\alpha_2(G'/G) + \alpha_0^2 \right) \\ & + bV \left(6\alpha_2(G'/G)^4 + (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) + \alpha_1\lambda\mu + 2\alpha_2\mu^2 \right) = 0 \end{aligned}$$

Or

$$\begin{aligned} & C - V \alpha_2(G'/G)^2 - V \alpha_1(G'/G) - V \alpha_0 + \alpha_2(G'/G)^2 \\ & + \alpha_1(G'/G) + \alpha_0 - a\alpha_2^2(G'/G)^4 - 2a\alpha_1\alpha_2(G'/G)^3 \\ & - a(\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 - 2a\alpha_1\alpha_2(G'/G) - a\alpha_0^2 \\ & + bV 6\alpha_2(G'/G)^4 + bV (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 \\ & + bV (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 \\ & + bV (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) \\ & + bV \alpha_1\lambda\mu + 2bV \alpha_2\mu^2 = 0 \end{aligned} \quad (1.51)$$

By equating each coefficient of (1.51) to zero, we have the system of algebraic equation as following

$$\begin{aligned}
C -V \alpha_0 + \alpha_0 - a\alpha_0^2 + bV \alpha_1 \lambda \mu + 2bV \alpha_2 \mu^2 &= 0, \\
-V \alpha_1 + \alpha_1 - 2a\alpha_1 \alpha_2 + 6bV \alpha_2 \lambda \mu + 2bV \alpha_1 \mu + bV \alpha_1 \lambda^2 &= 0, \\
-V \alpha_2 + \alpha_2 - a\alpha_1^2 - 2a\alpha_1 \alpha_2 + 8bV \alpha_2 \mu + 3bV \alpha_1 \lambda + 4bV \alpha_2 \lambda^2 &= 0, \\
-2a\alpha_1 \alpha_2 + 10bV \alpha_2 \lambda + 2bV \alpha_1 &= 0, \\
-a\alpha_2^2 + 6bV \alpha_2 &= 0,
\end{aligned} \tag{1.52}$$

and by solving the algebraic equations (1.52) for $\alpha_2, \alpha_1, \alpha_0, V, \lambda, \mu$ and C , we get

$$-a\alpha_2^2 + 6bV \alpha_2 = 0 \rightarrow a\alpha_2 = 6bV \rightarrow \alpha_2 = \frac{6bV}{a},$$

$$-2a\alpha_1 \left(\frac{6bV}{a}\right) + 10bV \left(\frac{6bV}{a}\right) \lambda + 2bV \alpha_1 = 0 \rightarrow \alpha_1 = \frac{6bV \lambda}{a}.$$

This leads to

$$V = \frac{-1}{6b \lambda^2 - 12b \lambda + 8b \mu - 1},$$

then,

$$\alpha_1 = \frac{-6b \lambda}{a(6b \lambda^2 - 12b \lambda + 8b \mu - 1)}, \quad \alpha_2 = \frac{-6b}{a(6b \lambda^2 - 12b \lambda + 8b \mu - 1)}$$

$$C = a\alpha_0^2 - \alpha_0 - \frac{\alpha_0}{6b \lambda^2 - 12b \lambda + 8b \mu - 1} - \frac{(6b^2 \lambda^2 \mu + 12b^2 \mu^2)}{a(6b \lambda^2 - 12b \lambda + 8b \mu - 1)^2},$$

and λ, μ and α_0 are arbitrary constants.

Then (1.47) be:

$$u(\zeta) = \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} (G'/G)^2 - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} (G'/G) + \alpha_0, \quad \zeta = x + \frac{t}{6b\lambda^2 - 12b\lambda + 8b\mu - 1}. \quad (1.53)$$

Case (I) $\lambda^2 - 4\mu = \omega_1^2 > 0$

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right)$$

$$G' = e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right)$$

$$(G'/G) = \frac{e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right)}$$

By using (1.23), so (G'/G) be:

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \frac{\left(c_1 \cosh \frac{\omega_1}{2}\xi + c_1 \sinh \frac{\omega_1}{2}\xi - c_2 \cosh \frac{\omega_1}{2}\xi + c_2 \sinh \frac{\omega_1}{2}\xi \right)}{\left(c_1 \cosh \frac{\omega_1}{2}\xi + c_1 \sinh \frac{\omega_1}{2}\xi + c_2 \cosh \frac{\omega_1}{2}\xi - c_2 \sinh \frac{\omega_1}{2}\xi \right)}.$$

Also, from (1.25) we get

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \frac{\left(B_1 \sinh \frac{\omega_1}{2}\xi + B_2 \cosh \frac{\omega_1}{2}\xi \right)}{\left(B_1 \cosh \frac{\omega_1}{2}\xi + B_2 \sinh \frac{\omega_1}{2}\xi \right)}.$$

By using (1.28) then (1.53) be

$$u(\zeta) = \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right)^2 - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right) + \alpha_0,$$

$$u(\zeta) = \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(\frac{\lambda^2}{4} + \frac{\omega_1^2}{4} K_1^2 - \frac{\lambda\omega_1}{2} K_1 \right) - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right) + \alpha_0,$$

or

$$u(\zeta) = \frac{-6b\lambda^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} - \frac{6b\omega_1^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_1^2 + \frac{6b\lambda\omega_1}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_1 + \frac{6b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} - \frac{6b\lambda\omega_1}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_1 + \alpha_0$$

Then

$$u(\zeta) = \frac{3b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \alpha_0 - \frac{6b\omega_1^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_1^2$$

So

$$u(\zeta) = \frac{3b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \alpha_0 - \frac{6b(\lambda^2 - 4\mu)}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_1^2 \quad (1.54)$$

Case (II) $\lambda^2 - 4\mu = -\omega_2 < 0$

For this case we have the solution

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{-\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{-\omega_2}}{2} \xi \right)$$

$$G' = e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{-\omega_2}}{2} c_1 \sin \frac{\sqrt{-\omega_2}}{2} \xi + \frac{\sqrt{-\omega_2}}{2} c_2 \cos \frac{\sqrt{-\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{-\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{-\omega_2}}{2} \xi \right).$$

$$(G'/G) = \frac{\left(e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{-\omega_2}}{2} c_1 \sin \frac{\sqrt{-\omega_2}}{2} \xi + \frac{\sqrt{-\omega_2}}{2} c_2 \cos \frac{\sqrt{-\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{-\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{-\omega_2}}{2} \xi \right) \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{-\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{-\omega_2}}{2} \xi \right)}.$$

If we set (1.25), then

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{-\omega_2}}{2} \left(\frac{B_2 \cos \frac{\sqrt{-\omega_2}}{2} \xi - B_1 \sin \frac{\sqrt{-\omega_2}}{2} \xi}{B_1 \cos \frac{\sqrt{-\omega_2}}{2} \xi + B_2 \sin \frac{\sqrt{-\omega_2}}{2} \xi} \right)$$

By using (1.37) then (1.53) be

$$u(\zeta) = \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\sqrt{-\omega_2}}{2} K_2 \right)^2 - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\sqrt{-\omega_2}}{2} K_2 \right) + \alpha_0$$

$$u(\zeta) = \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(\frac{\lambda^2}{4} - \frac{\lambda\sqrt{-\omega_2}}{2} K_2 - \frac{\omega_2}{4} K_2^2 \right) - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{\sqrt{-\omega_2}}{2} K_2 \right) + \alpha_0$$

Or

$$u(\zeta) = \frac{-6b\lambda^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \frac{6b\lambda\sqrt{-\omega_2}}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_2 + \frac{6b\omega_2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_2^2 + \frac{6b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} - \frac{6b\lambda\sqrt{-\omega_2}}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_2 + \alpha_0$$

So

$$u(\zeta) = \frac{6b\lambda^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \alpha_0 + \frac{6b\omega_2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_2^2$$

Then

$$u(\zeta) = \frac{3b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \alpha_0 - \frac{3b(\lambda^2 - 4\mu)}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} K_2^2 \quad (1.55)$$

Case (III) when $\lambda^2 - 4\mu = 0$

In this case we get two equal and real roots $m_1 = m_2$. For this case we have the solution

$$G = c_1 e^{-\frac{\lambda}{2}\xi} + c_2 \xi e^{-\frac{\lambda}{2}\xi}$$

So

$$G' = -\frac{\lambda}{2} c_1 e^{-\frac{\lambda}{2}\xi} + c_2 e^{-\frac{\lambda}{2}\xi} - \frac{\lambda}{2} c_2 \xi e^{-\frac{\lambda}{2}\xi}$$

Then

$$(G'/G) = \frac{-\frac{\lambda}{2} c_1 + c_2 - \frac{\lambda}{2} c_2 \xi}{c_1 + c_2 \xi}$$

Or

$$(G'/G) = -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi}$$

Then (1.53) be

$$\begin{aligned}
u(\zeta) &= \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \right)^2 \\
&\quad - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \right) + \alpha_0 \\
u(\zeta) &= \frac{-6b}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(\frac{\lambda^2}{4} - \frac{\lambda c_2}{c_1 + c_2\xi} + \frac{c_2^2}{(c_1 + c_2\xi)^2} \right) \\
&\quad - \frac{6b\lambda}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \right) + \alpha_0. \\
u(\zeta) &= \frac{-6b\lambda^2}{4a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \frac{6b\lambda c_2}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)(c_1 + c_2\xi)} \\
&\quad - \frac{6bc_2^2}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)(c_1 + c_2\xi)^2} + \frac{6b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} \\
&\quad - \frac{6b\lambda c_2}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)(c_1 + c_2\xi)} + \alpha_0. \\
u(\zeta) &= \frac{3b\lambda^2}{2a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)} + \alpha_0 \\
&\quad - \frac{6bc_2^2}{a(6b\lambda^2 - 12b\lambda + 8b\mu - 1)(c_1 + c_2\xi)^2}.
\end{aligned} \tag{1.56}$$

Chapter Two

The (KdV) Equation

The Kortweg-De Vries (KdV) equation is a nonlinear third order partial differential equation (Xian, 2015), which introduced by Joseph Boussinesq (1877). Then it revised by Diederik Korteweg and Gustav de Vries in (1895). In this chapter we introduce the general KdV equation, then, study the solution of the equation by using the (G'/G) -expansion method for $p = 1$, (Drazin & Johnson, 1983).

(2.1) The General KdV Equation

The general form of the general KdV equation is introduced by (Drazin & Johnson, 1983), in the form:

$$u_t + (p+1)(p+2)u^p u_x + u_{xxx} = 0, \quad (2.1)$$

where $p = 0, 1, 2, 3, \dots$ is an integer.

The general KdV equation (2.1) for $p = 0, 1, 2$ is classify as following:

(I) For $p = 0$, equation (2.1) is known as the linearized KdV equation and has the form

$$u_t + 2u_x + u_{xxx} = 0, \quad (2.2)$$

which is third order linear partial differential equation.

(II) For $p = 1$, equation (2.1) is known as the non-linear KdV equation and has the form

$$u_t + 6uu_x + u_{xxx} = 0, \quad (2.3)$$

(III) For $p = 2$, equation (2.1) is known as the modified KdV equation and has the form

$$u_t + 12u^2u_x + u_{xxx} = 0, \quad (2.4)$$

which, third order non-linear partial differential equations

Remark (2.1):

In general, the homogeneous balance number of the general KdV equation (2.1), is given by $(2/p)$.

(2.2). The Solution of the kdV Equation for $p = 1$

According to (2.3) and use the wave transform (1.2), the equation (2.3) can be reduce to the ordinary differential

$$-Vu' + 6uu' + u''' = 0. \quad (2.5)$$

The by integrating (2.5) with respect to ξ , we get

$$C -Vu + 3u^2 + u'' = 0. \quad (2.6)$$

Which has the homogeneous balance number $m = \frac{2}{p} = 2$. Now (1.4), reads

$$u(\xi) = \sum_{i=0}^2 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2 \quad (2.7)$$

To substitute (2.7) into (2.6), first we calculate

$$\begin{aligned}
u^2(\zeta) &= \left[\alpha_2 (G'/G)^2 + \alpha_1 (G'/G) + \alpha_0 \right]^2 \\
&= \alpha_2^2 (G'/G)^4 + 2\alpha_1\alpha_2 (G'/G)^3 + \alpha_1^2 (G'/G)^2 + \\
&\quad 2\alpha_1\alpha_2 (G'/G)^2 + 2\alpha_1\alpha_2 (G'/G) + \alpha_0^2,
\end{aligned}$$

which can be written in form:

$$\begin{aligned}
u^2(\zeta) &= \alpha_2^2 (G'/G)^4 + 2\alpha_1\alpha_2 (G'/G)^3 + (\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 \\
&\quad + 2\alpha_1\alpha_2 (G'/G) + \alpha_0^2
\end{aligned} \tag{2.8}$$

And by using (1.12) we can write u' as

$$\begin{aligned}
u' &= -2\alpha_2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 \\
&\quad - (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu
\end{aligned} \tag{2.9}$$

Then (2.9) and (1.12) give

$$\begin{aligned}
u'' &= 6\alpha_2 (G'/G)^4 + (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 + \\
&\quad (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 + \\
&\quad (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) + \alpha_1\lambda\mu + 2\alpha_2\mu^2.
\end{aligned} \tag{2.10}$$

Next, we substitute (2.7), (2.8) and (2.10) into (2.6), we get

$$\begin{aligned}
&C - V \left(\alpha_2 (G'/G)^2 + \alpha_1 (G'/G) + \alpha_0 \right) \\
&+ 3 \left(\alpha_2^2 (G'/G)^4 + 2\alpha_1\alpha_2 (G'/G)^3 + (\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 \right. \\
&\quad \left. + 2\alpha_1\alpha_2 (G'/G) + \alpha_0^2 \right) \\
&+ \left(6\alpha_2 (G'/G)^4 + (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 + \right. \\
&\quad \left. (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 + \right. \\
&\quad \left. (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) + \alpha_1\lambda\mu + 2\alpha_2\mu^2 \right) = 0,
\end{aligned} \tag{2.11}$$

Next, we expand (2.11) and collect the coefficients of G'/G , we can write

$$\begin{aligned}
& (3\alpha_2^2 + 6\alpha_2)(G'/G)^4 + (6\alpha_1\alpha_2 + 10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 \\
& + (-V\alpha_2 + 3(\alpha_1^2 + 2\alpha_1\alpha_2) + 8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 \\
& + (-V\alpha_1 + 6\alpha_1\alpha_2 + 6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) \\
& + C - V\alpha_0 + 3\alpha_0^2 + 2\alpha_2\mu^2 + \alpha_1\lambda\mu = 0.
\end{aligned} \tag{2.12}$$

This leads to the algebraic equations:

$$\begin{aligned}
C - V\alpha_0 + 3\alpha_0^2 + 2\alpha_2\mu^2 + \alpha_1\lambda\mu &= 0. \\
-V\alpha_1 + 6\alpha_1\alpha_2 + 6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2 &= 0. \\
-V\alpha_2 + 3(\alpha_1^2 + 2\alpha_1\alpha_2) + 8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2 &= 0. \\
6\alpha_1\alpha_2 + 10\alpha_2\lambda + 2\alpha_1 &= 0. \\
3\alpha_2^2 + 6\alpha_2 &= 0.
\end{aligned} \tag{2.13}$$

Next, we solving the algebraic equations (2.13) to get $\alpha_2, \alpha_1, \alpha_0, V, \lambda, \mu$ and C as following:

$$\begin{aligned}
3\alpha_2^2 + 6\alpha_2 = 0 &\rightarrow \alpha_2 = -2 \\
6\alpha_1(-2) + 10(-2)\lambda + 2\alpha_1 = 0 &\rightarrow -10\alpha_1 - 20\lambda = 0 \rightarrow \alpha_1 = -2\lambda \\
-V(-2\lambda) + 6(-2\lambda)(-2) + 6(-2)\lambda\mu + 2(-2\lambda)\mu + (-2\lambda)\lambda^2 = 0 \\
\rightarrow 2V\lambda + 24\lambda - 12\lambda\mu - 4\lambda\mu - 2\lambda^3 = 0 \\
\rightarrow V = \lambda^2 + 8\mu - 12 \\
C - (\lambda^2 + 8\mu - 12)\alpha_0 + 3\alpha_0^2 + 2(-2)\mu^2 + (-2\lambda)\lambda\mu = 0 \\
\rightarrow C - \lambda^2\alpha_0 - 8\mu\alpha_0 + 12\alpha_0 + 3\alpha_0^2 - 4\mu^2 - 2\lambda^2\mu = 0 \\
\rightarrow C = \alpha_0(\lambda^2 + 8\mu - 12 - 3\alpha_0) + \mu(4\mu + 2\lambda^2),
\end{aligned} \tag{2.14}$$

And λ, μ and α_0 are arbitrary constants.

Then (2.7) becomes

$$u(\zeta) = -2(G'/G)^2 - 2\lambda(G'/G) + \alpha_0, \quad \zeta = x - (\lambda^2 + 8\mu - 12)t \quad (2.15)$$

Now, the auxiliary equation of (1.5) has the roots (1.19).

For our equation we have the following cases:

Case (I) $\lambda^2 - 4\mu = \omega_1^2 > 0$, for this case, we have the solution

$$G = e^{-\frac{\lambda}{2}\zeta} \left(c_1 e^{\frac{\omega_1}{2}\zeta} + c_2 e^{-\frac{\omega_1}{2}\zeta} \right) \quad (2.16)$$

This gives

$$\begin{aligned} G' &= e^{-\frac{\lambda}{2}\zeta} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\zeta} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\zeta} \right) - \\ &\quad \frac{\lambda}{2} e^{-\frac{\lambda}{2}\zeta} \left(c_1 e^{\frac{\omega_1}{2}\zeta} + c_2 e^{-\frac{\omega_1}{2}\zeta} \right) \\ &= e^{-\frac{\lambda}{2}\zeta} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\zeta} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\zeta} - c_1 e^{\frac{\omega_1}{2}\zeta} - c_2 e^{-\frac{\omega_1}{2}\zeta} \right) \end{aligned} \quad (2.17)$$

The we can wite

$$(G'/G) = \frac{e^{-\frac{\lambda}{2}\zeta} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\zeta} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\zeta} - \frac{\lambda}{2} c_1 e^{\frac{\omega_1}{2}\zeta} - \frac{\lambda}{2} c_2 e^{-\frac{\omega_1}{2}\zeta} \right)}{e^{-\frac{\lambda}{2}\zeta} \left(c_1 e^{\frac{\omega_1}{2}\zeta} + c_2 e^{-\frac{\omega_1}{2}\zeta} \right)} \quad (2.18)$$

By using (1.23) and (1.25), we can write (2.18) as:

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{B_1 \sinh \frac{\omega_1}{2} \xi + B_2 \cosh \frac{\omega_1}{2} \xi}{B_1 \cosh \frac{\omega_1}{2} \xi + B_2 \sinh \frac{\omega_1}{2} \xi} \right) \quad (2.19)$$

Then, by substituting (1.28) into (2.19), then (2.7) becomes

$$\begin{aligned} u(\zeta) &= -2 \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right) + \alpha_0 \\ &= -2 \left(\frac{\lambda^2}{4} + \frac{\omega_1^2}{4} K_1^2 - \frac{\lambda \omega_1}{2} K_1 \right) - 2\lambda \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right) + \alpha_0 \end{aligned}$$

Then

$$u(\zeta) = \frac{\lambda^2}{2} + \alpha_0 - \frac{\omega_1^2}{2} K_1^2 \quad (2.20)$$

And, if we take, $B_1 \neq 0$, $B_2 = 0$, then (2.20) reads

$$\begin{aligned} u(\zeta) &= \frac{\lambda^2}{2} + \alpha_0 - \frac{\omega_1^2}{2} \left(\frac{\sinh \frac{\omega_1}{2} \xi}{\cosh \frac{\omega_1}{2} \xi} \right)^2 \\ &= \frac{\lambda^2}{2} + \alpha_0 - \frac{\omega_1^2}{2} \tanh^2 \frac{\omega_1}{2} \xi = \frac{\lambda^2}{2} + \alpha_0 - \frac{\omega_1^2}{2} \tanh^2 \frac{\omega_1}{2} \xi + 2\mu - 2\mu \end{aligned}$$

This gives

$$u(\zeta) = \alpha_0 + 2\mu + \frac{\omega_1^2}{2} \left(1 - \tanh^2 \frac{\omega_1}{2} \xi \right).$$

Then (2.20) reads

$$u(\xi) = \frac{(\lambda^2 - 4\mu)}{2} \left(\operatorname{sech}^2 \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) + \alpha_0 + 2\mu \quad (2.21)$$

which is equivalent to the solution of the solitary wave solution obtained by

Case (II) $\lambda^2 - 4\mu = -\omega_2 < 0$, for this case we have the solution

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right). \quad (2.22)$$

Also,

$$G' = e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right). \quad (2.23)$$

Then

$$(G'/G) = \frac{e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right)}. \quad (2.24)$$

Next, by using (1.25), can be written (2.24) as:

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{B_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - B_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{B_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + B_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right). \quad (2.25)$$

And by using (1.37) in (2.25). So can be written (2.15) as:

$$\begin{aligned} u(\zeta) &= -2 \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right) + \alpha_0 \\ &= -2 \left(\frac{\lambda^2}{4} - \frac{\lambda\sqrt{\omega_2}}{2} K_2 + \frac{\omega_2}{4} K_2^2 \right) - 2\lambda \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right) + \alpha_0. \end{aligned}$$

Then

$$u(\zeta) = \frac{\lambda^2}{2} + \alpha_0 - \frac{\omega_2}{2} K_2^2, \quad (2.26)$$

where, $\omega_2 = 4\mu - \lambda^2$. Hence (2.26) reads

$$\begin{aligned} u(\zeta) &= \frac{\lambda^2}{2} + \alpha_0 - \\ &\frac{(4\mu - \lambda^2)}{2} \left(\frac{B_2 \cos \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi - B_1 \sin \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi}{B_1 \cos \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi + B_2 \sin \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi} \right)^2. \end{aligned} \quad (2.27)$$

Case (III) $\lambda^2 - 4\mu = 0$, for this case we have the solution

$$G = c_1 e^{-\frac{\lambda}{2}\xi} + c_2 \xi e^{-\frac{\lambda}{2}\xi}. \quad (2.28)$$

This gives

$$G' = -\frac{\lambda}{2}c_1e^{-\frac{\lambda\xi}{2}} + c_2e^{-\frac{\lambda\xi}{2}} - \frac{\lambda}{2}c_2\xi e^{-\frac{\lambda\xi}{2}}. \quad (2.29)$$

Then

$$(G'/G) = \frac{-\frac{\lambda}{2}c_1 + c_2 - \frac{\lambda}{2}c_2\xi}{c_1 + c_2\xi} = -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi}. \quad (2.30)$$

Then (2.16) becomes:

$$\begin{aligned} u(\xi) &= -2\left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi}\right)^2 - 2\lambda\left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi}\right) + \alpha_0. \\ &= -\frac{\lambda^2}{2} - \frac{2c_2^2}{(c_1 + c_2\xi)^2} + \frac{2\lambda c_2}{c_1 + c_2\xi} + \lambda^2 - \frac{2\lambda c_2}{c_1 + c_2\xi} + \alpha_0. \end{aligned}$$

This gives

$$u(\xi) = \frac{\lambda^2}{2} + \alpha_0 - \frac{2c_2^2}{(c_1 + c_2\xi)^2} \quad (2.31)$$

At the end, we check, our result by using the Maple program tools, which we listed it Below:

```

> restart
> with(PDETools) :
> declare(u(x, t), u(xi), phi(xi), prime = xi)
      u(x, t) will now be displayed as u
      u(xi) will now be displayed as u
      phi(xi) will now be displayed as phi
derivatives with respect to xi of functions of one variable will now be displayed with ' (1)
>  $\frac{\partial}{\partial t} u(x, t) + 6 \cdot u(x, t) \left( \frac{\partial}{\partial x} u(x, t) \right) + \left( \frac{\partial^3}{\partial x^3} u(x, t) \right) = 0$ 
       $u_t + 6 u(x, t) u_x + u_{x,x,x} = 0$  (2)
> eval((2), u(x, t) = u(x - V·t))
-D(u)(-Vt + x) V + 6 u(-Vt + x) D(u)(-Vt + x) + D(3)(u)(-Vt + x) (3)
= 0
> eval((3), x - V·t = xi)
-D(u)(xi) V + 6 u D(u)(xi) + D(3)(u)(xi) = 0 (4)
> convert((4), diff)
      -u' V + 6 u u' + u''' = 0 (5)
> int(lhs((5)), xi) + C = 0
      -V u + 3 u2 + u'' + C = 0 (6)
>  $u(\text{xi}) = \sum_{i=0}^2 \alpha[i] \cdot \phi^i(\text{xi})$ 
       $u = \alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2$  (7)
> diff(phi(xi), xi) = -phi2(xi) - lambda·phi(xi) - mu
      phi' = -phi2 - lambda phi - mu (8)
> diff((7), xi)
      u' = alpha_1 phi' + 2 alpha_2 phi phi' (9)

```

> eval((9), (8))

$$u' = \alpha_1 (-\phi^2 - \lambda \phi - \mu) + 2 \alpha_2 \phi (-\phi^2 - \lambda \phi - \mu) \quad (10)$$

> expand((10))

$$u' = -\alpha_1 \phi^2 - \alpha_1 \lambda \phi - \alpha_1 \mu - 2 \alpha_2 \phi^3 - 2 \alpha_2 \phi^2 \lambda - 2 \alpha_2 \phi \mu \quad (11)$$

> diff((11), xi)

$$u'' = -2 \alpha_1 \phi \phi' - \alpha_1 \lambda \phi' - 6 \alpha_2 \phi^2 \phi' - 4 \alpha_2 \phi \lambda \phi' - 2 \alpha_2 \phi' \mu \quad (12)$$

> eval((12), (8))

$$u'' = -2 \alpha_1 \phi (-\phi^2 - \lambda \phi - \mu) - \alpha_1 \lambda (-\phi^2 - \lambda \phi - \mu) - 6 \alpha_2 \phi^2 (-\phi^2 - \lambda \phi - \mu) - 4 \alpha_2 \phi \lambda (-\phi^2 - \lambda \phi - \mu) - 2 \alpha_2 (-\phi^2 - \lambda \phi - \mu) \mu \quad (13)$$

> expand((13))

$$u'' = 2 \alpha_1 \phi^3 + 3 \alpha_1 \phi^2 \lambda + 2 \alpha_1 \phi \mu + \alpha_1 \lambda^2 \phi + \alpha_1 \lambda \mu + 6 \alpha_2 \phi^4 + 10 \alpha_2 \phi^3 \lambda + 8 \alpha_2 \phi^2 \mu + 4 \alpha_2 \phi^2 \lambda^2 + 6 \alpha_2 \phi \lambda \mu + 2 \alpha_2 \mu^2 \quad (14)$$

> eval((6), {(7), (14)})

$$-V (\alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2) + 3 (\alpha_0 + \alpha_1 \phi + \alpha_2 \phi^2)^2 + 2 \alpha_1 \phi^3 + 3 \alpha_1 \phi^2 \lambda + 2 \alpha_1 \phi \mu + \alpha_1 \lambda^2 \phi + \alpha_1 \lambda \mu + 6 \alpha_2 \phi^4 + 10 \alpha_2 \phi^3 \lambda + 8 \alpha_2 \phi^2 \mu + 4 \alpha_2 \phi^2 \lambda^2 + 6 \alpha_2 \phi \lambda \mu + 2 \alpha_2 \mu^2 + C = 0 \quad (15)$$

> expand((15))

$$-V \alpha_0 - V \alpha_1 \phi - V \alpha_2 \phi^2 + 3 \alpha_0^2 + 6 \alpha_0 \alpha_1 \phi + 6 \alpha_0 \alpha_2 \phi^2 + 3 \alpha_1^2 \phi^2 + 6 \alpha_1 \phi^3 \alpha_2 + 3 \alpha_2^2 \phi^4 + 2 \alpha_1 \phi^3 + 3 \alpha_1 \phi^2 \lambda + 2 \alpha_1 \phi \mu + \alpha_1 \lambda^2 \phi + \alpha_1 \lambda \mu + 6 \alpha_2 \phi^4 + 10 \alpha_2 \phi^3 \lambda + 8 \alpha_2 \phi^2 \mu + 4 \alpha_2 \phi^2 \lambda^2 + 6 \alpha_2 \phi \lambda \mu + 2 \alpha_2 \mu^2 + C = 0 \quad (16)$$

> collect({ (16)}, phi(xi))

$$\left\{ \begin{aligned} & (3\alpha_2^2 + 6\alpha_2)\phi^4 + (10\lambda\alpha_2 + 6\alpha_1\alpha_2 + 2\alpha_1)\phi^3 + (4\lambda^2\alpha_2 - V\alpha_2 + 3\lambda\alpha_1 \\ & + 8\mu\alpha_2 + 6\alpha_0\alpha_2 + 3\alpha_1^2)\phi^2 + (\lambda^2\alpha_1 + 6\lambda\mu\alpha_2 - V\alpha_1 + 2\alpha_1\mu \\ & + 6\alpha_0\alpha_1)\phi + \alpha_1\lambda\mu + 2\alpha_2\mu^2 - V\alpha_0 + 3\alpha_0^2 + C = 0 \end{aligned} \right\} \quad (17)$$

> { (3\alpha_2^2 + 6\alpha_2) = 0, (10\lambda\alpha_2 + 6\alpha_1\alpha_2 + 2\alpha_1) = 0, (4\lambda^2\alpha_2 - V\alpha_2 + 3\lambda\alpha_1 + 8\mu\alpha_2 + 6\alpha_0\alpha_2 + 3\alpha_1^2) = 0, (\lambda^2\alpha_1 + 6\lambda\mu\alpha_2 - V\alpha_1 + 2\mu\alpha_1 + 6\alpha_0\alpha_1) = 0, (\alpha_1\lambda\mu + 2\alpha_2\mu^2 - V\alpha_0 + 3\alpha_0^2 + C) = 0 }

$$\left\{ \begin{aligned} & 3\alpha_2^2 + 6\alpha_2 = 0, 10\lambda\alpha_2 + 6\alpha_1\alpha_2 + 2\alpha_1 = 0, \lambda^2\alpha_1 + 6\lambda\mu\alpha_2 - V\alpha_1 + 2\alpha_1\mu \\ & + 6\alpha_0\alpha_1 = 0, \alpha_1\lambda\mu + 2\alpha_2\mu^2 - V\alpha_0 + 3\alpha_0^2 + C = 0, 4\lambda^2\alpha_2 - V\alpha_2 \\ & + 3\lambda\alpha_1 + 8\mu\alpha_2 + 6\alpha_0\alpha_2 + 3\alpha_1^2 = 0 \end{aligned} \right\} \quad (18)$$

> solve((18), {alpha[0], alpha[1], alpha[2], C})

$$\left\{ \begin{aligned} & C = V\alpha_0 - 3\alpha_0^2, \alpha_0 = \alpha_0, \alpha_1 = 0, \alpha_2 = 0 \end{aligned} \right\}, \left\{ \begin{aligned} & C = \frac{2}{3}\lambda^2\mu - \frac{4}{3}\mu^2 + \frac{1}{12}V^2 \\ & - \frac{1}{12}\lambda^4, \alpha_0 = -\frac{\lambda^2}{6} + \frac{V}{6} - \frac{4\mu}{3}, \alpha_1 = -2\lambda, \alpha_2 = -2 \end{aligned} \right\} \quad (19)$$

> { C = \frac{2}{3}\lambda^2\mu - \frac{4}{3}\mu^2 + \frac{1}{12}V^2 - \frac{1}{12}\lambda^4, \alpha_0 = -\frac{\lambda^2}{6} + \frac{V}{6} - \frac{4\mu}{3}, \alpha_1 = -2\lambda, \alpha_2 = -2 }

$$\left\{ \begin{aligned} & C = \frac{2}{3}\lambda^2\mu - \frac{4}{3}\mu^2 + \frac{1}{12}V^2 - \frac{1}{12}\lambda^4, \alpha_0 = -\frac{\lambda^2}{6} + \frac{V}{6} - \frac{4\mu}{3}, \alpha_1 = -2\lambda, \\ & \alpha_2 = -2 \end{aligned} \right\} \quad (20)$$

Chapter Three

Boussinesq's Equations

The Boussinesq equation is simulating weakly nonlinear and long wave approximation that can be used in water waves, coastal engineering, and numerical models for water wave in harbors and shallow seas (**Himonas & Mantzavinos, 2015**). In this chapter we solve some forms of the Boussinesq's equation by using the (G'/G) -expansion method

(3.1). Boussinesq's equation

Example (3.1): We study the Boussinesq's equation in the form (**Himonas & Mantzavinos, 2015**).

$$u_{tt} - u_{xx} = u_{xxxx} + (u^2)_{xx}. \quad (3.1)$$

Use a transform (1.2), then (3.1) reduce to the ordinary differential equation:

$$V^2 u'' - u'' = u^{(4)} + (u^2)'' . \quad (3.2)$$

Or

$$(V^2 - 1)u'' = u^{(4)} + (u^2)'' .$$

By integration (3.2) with respect to ξ yields

$$C + V^2 u' - u' = u''' + (u^2)' .$$

Or

$$C + V^2 u' - u' - u''' - 2uu' = 0. \quad (3.3)$$

Considering the homogeneous balance number between u''' and uu' hence

$$(m + m + 1 = m + 3 \rightarrow m = 2)$$

For $m = 2$, (1.4) reads

$$u(\xi) = \sum_{i=0}^2 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2. \quad (3.4)$$

To substitute (3.4) into (3.3) we need

$$u' = -2\alpha_2 (G'/G)^3 - (2\alpha_2 \lambda + \alpha_1) (G'/G)^2 - (\alpha_1 \lambda + 2\alpha_2 \mu) (G'/G) - \alpha_1 \mu. \quad (3.5)$$

Then

$$u'' = 6\alpha_2 (G'/G)^4 + (10\alpha_2 \lambda + 2\alpha_1) (G'/G)^3 + (8\alpha_2 \mu + 3\alpha_1 \lambda + 4\alpha_2 \lambda^2) (G'/G)^2 + (6\alpha_2 \lambda \mu + 2\alpha_1 \mu + \alpha_1 \lambda^2) (G'/G) + \alpha_1 \lambda \mu + 2\alpha_2 \mu^2. \quad (3.6)$$

Then,

$$u''' = 24\alpha_2(G'/G)^3(G'/G)' + 3(10\alpha_2\lambda + 2\alpha_1)(G'/G)^2(G'/G)'$$

$$+ 2(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)(G'/G)' + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G)',$$

$$u''' = -24\alpha_2(G'/G)^5 - 24\alpha_2\lambda(G'/G)^4 - 24\alpha_2\mu(G'/G)^3$$

$$- 30\alpha_2\lambda(G'/G)^4 - 6\alpha_1(G'/G)^4 - 30\alpha_2\lambda^2(G'/G)^3$$

$$- 6\alpha_1\lambda(G'/G)^3 - 30\alpha_2\lambda\mu(G'/G)^2 - 6\alpha_1\mu(G'/G)^2$$

$$- 16\alpha_2\mu(G'/G)^3 - 6\alpha_1\lambda(G'/G)^3 - 8\alpha_2\lambda^2(G'/G)^3$$

$$- 16\alpha_2\mu\lambda(G'/G)^2 - 6\alpha_1\lambda^2(G'/G)^2 - 8\alpha_2\lambda^3(G'/G)^2$$

$$- 16\alpha_2\mu^2(G'/G) - 6\alpha_1\lambda\mu(G'/G) - 8\alpha_2\lambda^2\mu(G'/G)$$

$$- 6\alpha_2\lambda\mu(G'/G)^2 - 2\alpha_1\mu(G'/G)^2 - \alpha_1\lambda^2(G'/G)^2$$

$$- 6\alpha_2\lambda^2\mu(G'/G) - 2\alpha_1\mu\lambda(G'/G) - \alpha_1\lambda^3(G'/G)$$

$$- 6\alpha_2\lambda\mu^2 - 2\alpha_1\mu^2 - \alpha_1\lambda^2\mu$$

Or

$$u''' = -24\alpha_2(G'/G)^5 - (54\alpha_2\lambda + 6\alpha_1)(G'/G)^4$$

$$- (40\alpha_2\mu + 38\alpha_2\lambda^2 + 12\alpha_1\lambda)(G'/G)^3$$

$$- (52\alpha_2\lambda\mu + 8\alpha_1\mu + 7\alpha_1\lambda^2 + 8\alpha_2\lambda^3)(G'/G)^2$$

$$- (16\alpha_2\mu^2 + 8\alpha_1\lambda\mu + 14\alpha_2\lambda^2\mu + \alpha_1\lambda^3)(G'/G)$$

$$- 6\alpha_2\lambda\mu^2 - 2\alpha_1\mu^2 - \alpha_1\lambda^2\mu \quad (3.7)$$

Then (3.3) reads

$$\begin{aligned}
& C + V^2 \left(-2\alpha_2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 - (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu \right) \\
& - \left(-2\alpha_2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 - (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu \right) \\
& - \left(\begin{array}{l} -24\alpha_2 (G'/G)^5 - (54\alpha_2\lambda + 6\alpha_1)(G'/G)^4 \\ - (40\alpha_2\mu + 38\alpha_2\lambda^2 + 12\alpha_1\lambda)(G'/G)^3 \\ - (52\alpha_2\lambda\mu + 8\alpha_1\mu + 7\alpha_1\lambda^2 + 8\alpha_2\lambda^3)(G'/G)^2 \\ - (16\alpha_2\mu^2 + 8\alpha_1\lambda\mu + 14\alpha_2\lambda^2\mu + \alpha_1\lambda^3)(G'/G) \\ - 6\alpha_2\lambda\mu^2 - 2\alpha_1\mu^2 - \alpha_1\lambda^2\mu \end{array} \right) \\
& - 2 \left(\alpha_2 (G'/G)^2 + \alpha_1 (G'/G) + \alpha_0 \right) \left(\begin{array}{l} -2\alpha_2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 \\ - (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu \end{array} \right) = 0
\end{aligned}$$

Or

$$\begin{aligned}
& C - 2\alpha_2 V^2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1) V^2 (G'/G)^2 \\
& - (\alpha_1\lambda + 2\alpha_2\mu) V^2 (G'/G) - \alpha_1\mu V^2 + 2\alpha_2 (G'/G)^3 \\
& + (2\alpha_2\lambda + \alpha_1)(G'/G)^2 + (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) \\
& + \alpha_1\mu + 24\alpha_2 (G'/G)^5 + (54\alpha_2\lambda + 6\alpha_1)(G'/G)^4 \\
& + (40\alpha_2\mu + 38\alpha_2\lambda^2 + 12\alpha_1\lambda)(G'/G)^3 \\
& + (52\alpha_2\lambda\mu + 8\alpha_1\mu + 7\alpha_1\lambda^2 + 8\alpha_2\lambda^3)(G'/G)^2 \\
& + (16\alpha_2\mu^2 + 8\alpha_1\lambda\mu + 14\alpha_2\lambda^2\mu + \alpha_1\lambda^3)(G'/G) \\
& + 6\alpha_2\lambda\mu^2 + 2\alpha_1\mu^2 + \alpha_1\lambda^2\mu + 4\alpha_2^2 (G'/G)^5 \\
& + (4\alpha_2^2\lambda + 2\alpha_1\alpha_2)(G'/G)^4 + (2\alpha_1\alpha_2\lambda + 4\alpha_2^2\mu)(G'/G)^3 \\
& + 2\alpha_1\alpha_2\mu(G'/G)^2 + 4\alpha_1\alpha_2(G'/G)^4 + (4\alpha_1\alpha_2\lambda + 2\alpha_1^2)(G'/G)^3 \\
& + (2\alpha_1^2\lambda + 4\alpha_1\alpha_2\mu)(G'/G)^2 + 2\alpha_1^2\mu(G'/G) + 4\alpha_0\alpha_2(G'/G)^3 \\
& + (4\alpha_0\alpha_2\lambda + 2\alpha_0\alpha_1)(G'/G)^2 + (2\alpha_0\alpha_1\lambda + 4\alpha_0\alpha_2\mu)(G'/G) + 2\alpha_0\alpha_1\mu = 0.
\end{aligned}$$

$$\begin{aligned}
& (24\alpha_2 + 4\alpha_2^2)(G'/G)^5 + (54\alpha_2\lambda + 6\alpha_1 + 4\alpha_2^2\lambda + 2\alpha_1\alpha_2 + 4\alpha_1\alpha_2)(G'/G)^4 \\
& + \left(\begin{array}{l} -2\alpha_2V^2 + 2\alpha_2 + 40\alpha_2\mu + 38\alpha_2\lambda^2 + 12\alpha_1\lambda + \\ 2\alpha_1\alpha_2\lambda + 4\alpha_2^2\mu + 4\alpha_1\alpha_2\lambda + 2\alpha_1^2 + 4\alpha_0\alpha_2 \end{array} \right) (G'/G)^3 \\
& + \left(\begin{array}{l} -2\alpha_2\mathcal{V}^2 - \alpha_1V^2 + 2\alpha_2\lambda + \alpha_1 + 52\alpha_2\lambda\mu + 8\alpha_1\mu + 7\alpha_1\lambda^2 \\ + 8\alpha_2\lambda^3 + 2\alpha_1\alpha_2\mu + 4\alpha_0\alpha_2\lambda + 2\alpha_0\alpha_1 + 2\alpha_1^2\lambda + 4\alpha_1\alpha_2\mu \end{array} \right) (G'/G)^2 \quad (3.8) \\
& + \left(\begin{array}{l} -\alpha_1\mathcal{V}^2 - 2\alpha_2\mu\mathcal{V}^2 + \alpha_1\lambda + 2\alpha_2\mu + 16\alpha_2\mu^2 + 8\alpha_1\lambda\mu \\ + 14\alpha_2\lambda^2\mu + \alpha_1\lambda^3 + 2\alpha_1^2\mu + 2\alpha_0\alpha_1\lambda + 4\alpha_0\alpha_2\mu \end{array} \right) (G'/G) \\
& + C - \alpha_1\mu\mathcal{V}^2 + \alpha_1\mu + 6\alpha_2\lambda\mu^2 + 2\alpha_1\mu^2 + \alpha_1\lambda^2\mu + 2\alpha_0\alpha_1\mu = 0,
\end{aligned}$$

By equating each coefficient of (3.8) to zero yields

$$24\alpha_2 + 4\alpha_2^2 = 0$$

$$54\alpha_2\lambda + 6\alpha_1 + 4\alpha_2^2\lambda + 6\alpha_1\alpha_2 = 0$$

$$\begin{aligned}
-2\alpha_2V^2 + 2\alpha_2 + 40\alpha_2\mu + 38\alpha_2\lambda^2 + 12\alpha_1\lambda + 6\alpha_1\alpha_2\lambda \\
+ 4\alpha_2^2\mu + 2\alpha_1^2 + 4\alpha_0\alpha_2 = 0
\end{aligned}$$

$$\begin{aligned}
-2\alpha_2\mathcal{V}^2 + \alpha_1V^2 + 2\alpha_2\lambda + \alpha_1 + 52\alpha_2\lambda\mu + 8\alpha_1\mu + 7\alpha_1\lambda^2 + \\
8\alpha_2\lambda^3 + 2\alpha_1\alpha_2\mu + 4\alpha_0\alpha_2\lambda + 2\alpha_0\alpha_1 + 2\alpha_1^2\lambda + 4\alpha_1\alpha_2\mu = 0
\end{aligned}$$

$$\begin{aligned}
-\alpha_1\mathcal{V}^2 - 2\alpha_2\mu\mathcal{V}^2 + \alpha_1\lambda + 2\alpha_2\mu + 16\alpha_2\mu^2 + 8\alpha_1\lambda\mu \\
+ 14\alpha_2\lambda^2\mu + \alpha_1\lambda^3 + 2\alpha_1^2\mu + 2\alpha_0\alpha_1\lambda + 4\alpha_0\alpha_2\mu = 0
\end{aligned}$$

$$C - \alpha_1\mu\mathcal{V}^2 + \alpha_1\mu + 6\alpha_2\lambda\mu^2 + 2\alpha_1\mu^2 + \alpha_1\lambda^2\mu + 2\alpha_0\alpha_1\mu = 0. \quad (3.9)$$

Solving the algebraic equations above to get $\alpha_2, \alpha_1, \alpha_0, V, \lambda, \mu$ and C as follows:

$$24\alpha_2 + 4\alpha_2^2 = 0 \rightarrow \alpha_2 = \frac{-24}{4} \rightarrow \alpha_2 = -6$$

$$\begin{aligned} 54(-6)\lambda + 6\alpha_1 + 4(-6)^2\lambda + 6\alpha_1(-6) &= 0 \\ \rightarrow -30\alpha_1 - 180\lambda &= 0 \\ \rightarrow \alpha_1 &= -6\lambda \end{aligned}$$

$$\begin{aligned} -2(-6)V^2 + 2(-6) + 40(-6)\mu + 38(-6)\lambda^2 + 12(-6\lambda)\lambda + \\ 6(-6\lambda)(-6)\lambda + 4(-6)^2\mu + 2(-6\lambda)^2 + 4\alpha_0(-6) &= 0 \\ \rightarrow 12V^2 - 12\lambda^2 - 96\mu - 24\alpha_0 - 12 &= 0 \\ \rightarrow V &= \sqrt{(\lambda^2 + 8\mu + 2\alpha_0 + 1)} \end{aligned}$$

$$\begin{aligned} C - (-6\lambda)\left(\sqrt{(\lambda^2 + 8\mu + 2\alpha_0 + 1)}\right)^2 \mu + (-6\lambda)\mu + 6(-6)\lambda\mu^2 + \\ 2(-6\lambda)\mu^2 + (-6\lambda)\lambda^2\mu + 2\alpha_0(-6\lambda)\mu &= 0 \\ \rightarrow C &= 0 \end{aligned} \tag{3.10}$$

Where λ, μ and α_0 are arbitrary constants.

$$u(\xi) = -6(G'/G)^2 - 6\lambda(G'/G) + \alpha_0, \tag{3.11}$$

$$\xi = x - \left(\sqrt{(\lambda^2 + 8\mu + 2\alpha_0 + 1)}\right)t$$

Now, the auxiliary equation of (1.5) has the roots (1.19)

For solution we have these cases

Case (I) $\lambda^2 - 4\mu = \omega_1^2 > 0$

Using (1.26) and (1.28) then (3.11) be

$$u(\xi) = -6\left(-\frac{\lambda}{2} + \frac{\omega_1}{2}K_1\right)^2 - 6\lambda\left(-\frac{\lambda}{2} + \frac{\omega_1}{2}K_1\right) + \alpha_0$$

$$u(\xi) = -6 \left(\frac{\lambda^2}{4} + \frac{\omega_1^2}{4} K_1^2 - \frac{\lambda \omega_1}{2} K_1 \right) - 6\lambda \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} K_1 \right) + \alpha_0$$

Then

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{3\omega_1^2}{2} K_1^2 \quad (3.12)$$

If we take $B_1 \neq 0$, $B_2 = 0$, $\mu = 0$ then

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{3\lambda^2}{2} \left(\frac{\sinh \frac{\lambda}{2} \xi}{\cosh \frac{\lambda}{2} \xi} \right)^2$$

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{3\lambda^2}{2} \tanh^2 \frac{\lambda}{2} \xi$$

$$u(\xi) = \alpha_0 + \frac{3\lambda^2}{2} \left(1 - \tanh^2 \frac{\lambda}{2} \xi \right)$$

Where

$$\operatorname{sech}^2 \frac{\lambda}{2} \xi = \left(1 - \tanh^2 \frac{\lambda}{2} \xi \right)$$

Then (3.12) reads

$$u(\xi) = \alpha_0 + \frac{3\lambda^2}{2} \left(\operatorname{sech}^2 \frac{\lambda}{2} \xi \right) \quad (3.13)$$

The solution (3.13) is equivalent to the solution obtained by He's semi inverse process to find the solitary wave solution (Almatrafi, Alharbi & Tunç,2020).

Case (II) $\lambda^2 - 4\mu = -\omega_2 < 0$,

By using (1.34) and (1.37), can be written (3.11) as

$$u(\xi) = -6 \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right)^2 - 6\lambda \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right) + \alpha_0$$

Or

$$u(\xi) = -6 \left(\frac{\lambda^2}{4} - \frac{\lambda\sqrt{\omega_2}}{2} K_2 + \frac{\omega_2}{4} K_2^2 \right) - 6\lambda \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} K_2 \right) + \alpha_0$$

Then

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{3\omega_2}{2} K_2^2$$

Hence $-\omega_2 = \lambda^2 - 4\mu \rightarrow \omega_2 = 4\mu - \lambda^2$

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{3(4\mu - \lambda^2)}{2} \left(\frac{B_2 \cos \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi - B_1 \sin \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi}{B_1 \cos \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi + B_2 \sin \frac{\sqrt{(4\mu - \lambda^2)}}{2} \xi} \right)^2 \quad (3.14)$$

Case (III) when $\lambda^2 - 4\mu = 0$

Using (1.39) then (3.11) reads

$$u(\xi) = -6 \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right)^2 - 6\lambda \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right) + \alpha_0$$

Or

$$u(\xi) = -\frac{6\lambda^2}{4} - \frac{6c_2^2}{(c_1 + c_2\xi)^2} + \frac{6\lambda c_2}{c_1 + c_2\xi} + \frac{6\lambda^2}{2} - \frac{6\lambda c_2}{c_1 + c_2\xi} + \alpha_0$$

Then

$$u(\xi) = \frac{3\lambda^2}{2} + \alpha_0 - \frac{6c_2^2}{(c_1 + c_2\xi)^2} \quad \text{where } \xi = x - \sqrt{(\lambda^2 + 8\mu + 2\alpha_0 + 1)} t \quad (3.15)$$

(3.2) Variant Boussinesq's equation:

Consider the variant Boussinesq's equation which is known as a model for water waves in the form (Wang, Li & Zhang, 2008).

$$H_t + (Hu)_x + u_{xxx} = 0 \quad (3.16)$$

$$u_t + H_x + uu_x = 0 \quad (3.17)$$

Where $H = H(x, t)$ is total depth, and $u = u(x, t)$ is velocity which is the speed with given direction

Suppose the transforms

$$H(x, t) = H(\xi), \quad \xi = x - Vt \quad (3.18)$$

$$u(x, t) = u(\xi), \quad \xi = x - Vt \quad (3.19)$$

Using transforms (3.18) and (3.19), then (3.16) (3.17) reduce to the ordinary differential equation

$$-VH' + (Hu)' + u''' = 0. \quad (3.20)$$

$$-Vu' + H' + uu' = 0. \quad (3.21)$$

Then by integrating , will reduce to

$$C_1 -VH + Hu + u'' = 0. \quad (3.22)$$

$$C_2 -Vu + H + \frac{1}{2}u^2 = 0. \quad (3.23)$$

Where C_1 and C_2 are integration constants

Consider a homogeneous balance between u'' and Hu and H and u^2 . Hence homogeneous balance number between u'' and Hu

$$(m_1 + m_2 = m_2 + 2) \text{ and } (2m_2 = m_1) \text{ then } m_1 = 2 \quad \text{and} \quad m_2 = 1$$

So, for $m = 2$ and $m = 1$, (3.24) and (3.25) reads

$$H(\xi) = \sum_{i=0}^2 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2. \quad (3.24)$$

$$u(\xi) = \sum_{i=0}^1 \beta_i (G'/G)^i = \beta_0 + \beta_1 (G'/G). \quad (3.25)$$

To substitute (3.24) into (3.22), we need

$$Hu = \alpha_2 \beta_1 (G'/G)^3 + (\alpha_1 \beta_1 + \alpha_2 \beta_0)(G'/G)^2 + (\alpha_1 \beta_0 + \alpha_0 \beta_1)(G'/G) + \alpha_0 \beta_0. \quad (3.26)$$

And

$$u'' = 2\beta_1(G'/G)^3 + 3\beta_1\lambda(G'/G)^2 + (2\beta_1\mu + \beta_1\lambda^2)(G'/G) + \beta_1\lambda\mu. \quad (3.27)$$

To substitute (3.25) into (3.23), we need

$$u^2 = \beta_1^2(G'/G)^2 + 2\beta_1\beta_0(G'/G) + \beta_0^2. \quad (3.28)$$

Then we obtain

$$C_1 - V \left(\alpha_2(G'/G)^2 + \alpha_1(G'/G) + \alpha_0 \right) + \left(\begin{array}{l} \alpha_2\beta_1(G'/G)^3 + (\alpha_1\beta_1 + \alpha_2\beta_0)(G'/G)^2 + \\ (\alpha_1\beta_0 + \alpha_0\beta_1)(G'/G) + \alpha_0\beta_0 \end{array} \right) + \left(\begin{array}{l} 2\beta_1(G'/G)^3 + 3\beta_1\lambda(G'/G)^2 + \\ (2\beta_1\mu + \beta_1\lambda^2)(G'/G) + \beta_1\lambda\mu \end{array} \right) = 0,$$

This read

$$\begin{aligned} & (\alpha_2\beta_1 + 2\beta_1)(G'/G)^3 + (-V\alpha_2 + \alpha_1\beta_1 + \alpha_2\beta_0 + 3\beta_1\lambda)(G'/G)^2 \\ & + (-V\alpha_1 + \alpha_1\beta_0 + \alpha_0\beta_1 + 2\beta_1\mu + \beta_1\lambda^2)(G'/G) \\ & + C_1 - V\alpha_0 + \alpha_0\beta_0 + \beta_1\lambda\mu = 0. \end{aligned} \quad (3.29)$$

And,

$$\begin{aligned} & C_2 - V(\beta_1(G'/G) + \beta_0) + (\alpha_2(G'/G)^2 + \alpha_1(G'/G) + \alpha_0) \\ & + \frac{1}{2}(\beta_1^2(G'/G)^2 + 2\beta_1\beta_0(G'/G) + \beta_0^2) = 0. \end{aligned}$$

$$\begin{aligned} & \left(\alpha_2 + \frac{1}{2} \beta_1^2 \right) (G'/G)^2 + (-V \beta_1 + \alpha_1 + \beta_1 \beta_0) (G'/G) \\ & + C_2 - V \beta_0 + \alpha_0 + \frac{1}{2} \beta_0^2 = 0. \end{aligned} \quad (3.30)$$

Equating each coefficients of (3.29) and (3.30) to zero yields the system of algebraic equations

$$\begin{cases} \alpha_2 \beta_1 + 2\beta_1 = 0 \\ -V \alpha_2 + \alpha_1 \beta_1 + \alpha_2 \beta_0 + 3\beta_1 \lambda = 0 \\ -V \alpha_1 + \alpha_1 \beta_0 + \alpha_0 \beta_1 + 2\beta_1 \mu + \beta_1 \lambda^2 = 0 \\ C_1 - V \alpha_0 + \alpha_0 \beta_0 + \beta_1 \lambda \mu = 0 \end{cases} \quad (3.31-a)$$

$$\begin{cases} \alpha_2 + \frac{1}{2} \beta_1^2 = 0 \\ -V \beta_1 + \alpha_1 + \beta_1 \beta_0 = 0 \\ C_2 - V \beta_0 + \alpha_0 + \frac{1}{2} \beta_0^2 = 0 \end{cases} \quad (3.31-b)$$

Solving the algebraic equations above to get $\alpha_2, \alpha_1, \alpha_0, V, \lambda, \mu$ and C as follows

Now from the first equation of (3.31-a) we can calculate $\alpha_2 = -2$ and from the first equation of (3.31-b) we can calculate $\beta_1 = \pm 2$.

Also,

$$\begin{aligned}
& -V\alpha_2 + \alpha_1\beta_1 + \alpha_2\beta_0 + 3\beta_1\lambda = 0 \\
& \rightarrow 2V + 2\alpha_1 - 2\beta_0 + 6\lambda = 0 \\
& \rightarrow V + \alpha_1 = \beta_0 - 3\lambda \rightarrow (1)
\end{aligned}$$

$$\begin{aligned}
& -V\beta_1 + \alpha_1 + \beta_1\beta_0 = 0 \\
& \rightarrow -2V + \alpha_1 + 2\beta_0 = 0 \\
& \rightarrow -2V + \alpha_1 = -2\beta_0 \rightarrow (2) \\
& \text{from (1) \& (2)} \rightarrow V = \beta_0 \mp \lambda
\end{aligned}$$

$$\begin{aligned}
& -(\beta_0 \mp \lambda)(-2) + 2\alpha_1 - 2\beta_0 + 6\lambda = 0 \\
& \rightarrow \alpha_1 = -2\lambda
\end{aligned}$$

$$\begin{aligned}
& -V\alpha_1 + \alpha_1\beta_0 + \alpha_0\beta_1 + 2\beta_1\mu + \beta_1\lambda^2 = 0 \\
& \rightarrow (-\beta_0 + \lambda)(-2\lambda) - 2\lambda\beta_0 + 2\alpha_0 + 4\mu + 2\lambda^2 = 0 \\
& \rightarrow \alpha_0 = -2\mu
\end{aligned}$$

$$\begin{aligned}
& C_1 - V\alpha_0 + \alpha_0\beta_0 + \beta_1\lambda\mu = 0 \\
& \rightarrow C_1 - (\beta_0 - \lambda)(-2\mu) + \beta_0(-2\mu) + 2\lambda\mu = 0 \\
& \rightarrow C_1 = 0 \tag{3.32}
\end{aligned}$$

$$\begin{aligned}
& C_2 - V\beta_0 + \alpha_0 + \frac{1}{2}\beta_0^2 = 0 \\
& \rightarrow C_2 - (\beta_0 - \lambda)\beta_0 + (-2\mu) + \frac{1}{2}\beta_0^2 = 0 \\
& \rightarrow C_2 = \frac{1}{2}\beta_0^2 + 2\mu \mp \lambda\beta_0
\end{aligned}$$

Where λ , μ and α_0 are arbitrary constants.

Then (3.24) and (3.25) becomes

$$H(\xi) = -2(G'/G)^2 - 2\lambda(G'/G) - 2\mu, \quad \xi = x - (\beta_0 \mp \lambda)t \tag{3.33}$$

$$u(\xi) = \pm 2(G'/G) + \beta_0, \quad (3.34)$$

Now, we have the following cases

$$\text{Case (I)} \quad \lambda^2 - 4\mu = \omega_1^2 > 0$$

For this case has a solution

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right) \quad (3.35)$$

So

$$G' = e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right) \quad (3.36)$$

Then

$$(G'/G) = \frac{e^{-\frac{\lambda}{2}\xi} \left(\frac{\omega_1}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\omega_1}{2} c_2 e^{-\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_1 e^{\frac{\omega_1}{2}\xi} - \frac{\lambda}{2} c_2 e^{-\frac{\omega_1}{2}\xi} \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 e^{\frac{\omega_1}{2}\xi} + c_2 e^{-\frac{\omega_1}{2}\xi} \right)} \quad (3.37)$$

By using (1.23) so (3.37) can be written as

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{c_1 \cosh \frac{\omega_1}{2} \xi + c_1 \sinh \frac{\omega_1}{2} \xi - c_2 \cosh \frac{\omega_1}{2} \xi + c_2 \sinh \frac{\omega_1}{2} \xi}{c_1 \cosh \frac{\omega_1}{2} \xi + c_1 \sinh \frac{\omega_1}{2} \xi + c_2 \cosh \frac{\omega_1}{2} \xi - c_2 \sinh \frac{\omega_1}{2} \xi} \right) \quad (3.38)$$

Suppose

$$A_1 = c_1 + c_2, \quad A_2 = c_1 - c_2 \quad (3.39)$$

then, (3.38) can be written in the form

$$(G'/G) = -\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{A_1 \sinh \frac{\omega_1}{2} \xi + A_2 \cosh \frac{\omega_1}{2} \xi}{A_1 \cosh \frac{\omega_1}{2} \xi + A_2 \sinh \frac{\omega_1}{2} \xi} \right) \quad (3.40)$$

Then (3.33) be

$$H(\xi) = -2 \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{A_1 \sinh \frac{\omega_1}{2} \xi + A_2 \cosh \frac{\omega_1}{2} \xi}{A_1 \cosh \frac{\omega_1}{2} \xi + A_2 \sinh \frac{\omega_1}{2} \xi} \right) \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{A_1 \sinh \frac{\omega_1}{2} \xi + A_2 \cosh \frac{\omega_1}{2} \xi}{A_1 \cosh \frac{\omega_1}{2} \xi + A_2 \sinh \frac{\omega_1}{2} \xi} \right) \right) - 2\mu$$

$$H(\xi) = -\frac{\omega_1^2}{2} \left(\frac{A_1 \sinh \frac{\omega_1}{2} \xi + A_2 \cosh \frac{\omega_1}{2} \xi}{A_1 \cosh \frac{\omega_1}{2} \xi + A_2 \sinh \frac{\omega_1}{2} \xi} \right)^2 + \frac{\lambda^2}{2} - 2\mu$$

Or

$$H(\xi) = \frac{\lambda^2}{2} - 2\mu - \frac{\lambda^2 - 4\mu}{2} \left(\frac{A_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + A_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{A_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + A_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right)^2 \quad (3.41)$$

Also,(3.34) be

$$\begin{aligned}
u(\xi) &= \pm 2 \left(-\frac{\lambda}{2} + \frac{\omega_1}{2} \left(\frac{A_1 \sinh \frac{\omega_1}{2} \xi + A_2 \cosh \frac{\omega_1}{2} \xi}{A_1 \cosh \frac{\omega_1}{2} \xi + A_2 \sinh \frac{\omega_1}{2} \xi} \right) \right) + \beta_0 \\
&= \mp \lambda + \sqrt{\lambda^2 - 4\mu} \left(\frac{A_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + A_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{A_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + A_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right) + \beta_0
\end{aligned} \tag{3.42}$$

Case (II) $\lambda^2 - 4\mu = -\omega_2 < 0$,

For this case we have the solution

$$G = e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right) \tag{3.45}$$

Also,

$$\begin{aligned}
G' &= e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \\
&\frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right)
\end{aligned} \tag{3.46}$$

Then

$$(G'/G) = \frac{\left(e^{-\frac{\lambda}{2}\xi} \left(-\frac{\sqrt{\omega_2}}{2} c_1 \sin \frac{\sqrt{\omega_2}}{2} \xi + \frac{\sqrt{\omega_2}}{2} c_2 \cos \frac{\sqrt{\omega_2}}{2} \xi \right) - \frac{\lambda}{2} e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right) \right)}{e^{-\frac{\lambda}{2}\xi} \left(c_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + c_2 \sin \frac{\sqrt{\omega_2}}{2} \xi \right)} \quad (3.47)$$

By using (3.39), can be written (3.47) as

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{A_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - A_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{A_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + A_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) \quad (3.48)$$

Then (3.33) be

$$H(\xi) = -2 \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{A_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - A_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{A_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + A_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{A_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - A_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{A_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + A_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) \right) - 2\mu$$

Or

$$H(\xi) = -\frac{\omega_2}{4} \left(\frac{A_1 \sinh \frac{\omega_2}{2} \xi + A_2 \cosh \frac{\omega_2}{2} \xi}{A_1 \cosh \frac{\omega_2}{2} \xi + A_2 \sinh \frac{\omega_2}{2} \xi} \right)^2 + \frac{\lambda^2}{2} - 2\mu$$

$$H(\xi) = -\frac{\omega_2}{4} \left(\frac{A_1 \sinh \frac{\omega_2}{2} \xi + A_2 \cosh \frac{\omega_2}{2} \xi}{A_1 \cosh \frac{\omega_2}{2} \xi + A_2 \sinh \frac{\omega_2}{2} \xi} \right)^2 + \frac{\lambda^2}{2} - 2\mu$$

$$H(\xi) = \frac{\lambda^2}{2} - 2\mu - \frac{(4\mu - \lambda^2)}{4} \left(\frac{A_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi - A_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi}{A_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + A_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi} \right)^2 \quad (3.49)$$

Furthermore, (3.34) reads

$$u(\xi) = \pm 2 \left(-\frac{\lambda}{2} + \frac{\sqrt{\omega_2}}{2} \left(\frac{A_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - A_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{A_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + A_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) \right) + \beta_0$$

Or

$$u(\xi) = \sqrt{\omega_2} \left(\frac{A_2 \cos \frac{\sqrt{\omega_2}}{2} \xi - A_1 \sin \frac{\sqrt{\omega_2}}{2} \xi}{A_1 \cos \frac{\sqrt{\omega_2}}{2} \xi + A_2 \sin \frac{\sqrt{\omega_2}}{2} \xi} \right) + \beta_0 \mp \lambda$$

$$u(\xi) = \sqrt{4\mu - \lambda^2} \left(\frac{A_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi - A_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi}{A_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + A_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi} \right) + \beta_0 \mp \lambda \quad (3.50)$$

Case (III) when $\lambda^2 - 4\mu = 0$

So the two equal and real roots $m_1 = m_2$. For this case we have the solution

$$G = c_1 e^{-\frac{\lambda}{2}\xi} + c_2 \xi e^{-\frac{\lambda}{2}\xi} \quad (3.51)$$

And

$$G' = -\frac{\lambda}{2} c_1 e^{-\frac{\lambda}{2}\xi} + c_2 e^{-\frac{\lambda}{2}\xi} - \frac{\lambda}{2} c_2 \xi e^{-\frac{\lambda}{2}\xi} \quad (3.52)$$

So

$$(G'/G) = \frac{-\frac{\lambda}{2} c_1 + c_2 - \frac{\lambda}{2} c_2 \xi}{c_1 + c_2 \xi}$$

Or

$$(G'/G) = -\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \quad (3.53)$$

Then (3.33) be

$$H(\xi) = -2 \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right) - 2\mu$$

Or

$$H(\xi) = -2 \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right)^2 - 2\lambda \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2 \xi} \right) - 2\mu$$

$$H(\xi) = -\frac{\lambda^2}{2} - \frac{2c_2^2}{(c_1 + c_2\xi)^2} + \frac{2\lambda c_2}{c_1 + c_2\xi} + \lambda^2 - \frac{2\lambda c_2}{c_1 + c_2\xi} - 2\mu$$

$$H(\xi) = -\frac{2c_2^2}{(c_1 + c_2\xi)^2} \tag{3.54}$$

Also, (3.34) can be written

$$u(\xi) = \pm 2 \left(-\frac{\lambda}{2} + \frac{c_2}{c_1 + c_2\xi} \right) + \beta_0$$

Or

$$u(\xi) = \pm \frac{2c_2}{c_1 + c_2\xi} + \beta_0 \mp \lambda \tag{3.55}$$

Chapter Four

The General KdV Equation for $p = 3$

(4.1) The solution of the general KdV equation for $p = 3$

In chapter two, we introduced the general KdV equation and study the solutions for $p = 1$, which has a positive integer homogeneous balance number. In this chapter, we study the solution of general KdV equation for $p = 3$ (**Drazin & Johnson, 1983**), which has the form

$$u_t + 20u^3u_x + u_{xxx} = 0 \quad (4.1)$$

By using the wave transform (1.2), we can reduce (4.1) to an ordinary differential equation of the form

$$-Vu' + 20u^3u' + u''' = 0 \quad (4.2)$$

which we can integrate with respect to ξ , to get the second order ordinary differential equation

$$C -Vu + 5u^4 + u'' = 0 \quad (4.3)$$

Next we consider the homogeneous balance between of the terms u'' and u^4 . This gives $(4m = m + 2)$, which leads to non-integer homogeneous balance number $m = 2/3$. To reduce (4.3) to an equation with positive homogeneous balance number, we follow (**Zayed & Alurfi, 2014**) and assume

$$u(\xi) = \Phi^{\frac{2}{3}}(\xi) \quad (4.4)$$

Next substituting (4.4) into (4.3) implies

$$C - V(\Phi^{\frac{2}{3}}) + 5\Phi^{\frac{8}{3}} - \frac{2(\Phi')^2}{9\Phi^{\frac{4}{3}}} + \frac{2(\Phi'')}{3\Phi^{\frac{1}{3}}} = 0. \quad (4.5)$$

Then by multiply the both sides of (4.5) by $\Phi^{\frac{8}{3}}$, we can write

$$180\Phi^4 - 36V(\Phi^2) + 36C(\Phi^{\frac{4}{3}}) + 24(\Phi'')(\Phi) - 8(\Phi')^2 = 0 \quad (4.6)$$

Hence (4.6) can be integrating if $C = 0$, this leads to the ordinary differantion equation

$$45\Phi^4 - 9V(\Phi^2) + 6(\Phi'')(\Phi) - 2(\Phi')^2 = 0 \quad (4.7)$$

which has homogeneous balance number $m = 1$. Now we can assume

$$\Phi(\xi) = \sum_{i=0}^1 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) \quad (4.8)$$

To substitute (4.8) into (4.7) we need

$$\Phi' = -\alpha_1 (G'/G)^2 - \alpha_1 \lambda (G'/G) - \alpha_1 \mu \quad (4.9)$$

Then

$$\Phi'' = 2\alpha_1 (G'/G)^3 + 3\alpha_1 \lambda (G'/G)^2 + (2\alpha_1 \mu + \alpha_1 \lambda^2)(G'/G) + \alpha_1 \lambda \mu \quad (4.10)$$

By using (4.8) we can find

$$\Phi^2 = [\alpha_1 (G'/G) + \alpha_0]^2 = \alpha_1^2 (G'/G)^2 + 2\alpha_1 \alpha_0 (G'/G) + \alpha_0^2 \quad (4.11)$$

$$\begin{aligned}\Phi^4 = & \alpha_1^4 (G'/G)^4 + 4\alpha_1^3 \alpha_0 (G'/G)^3 + \\ & 6\alpha_1^2 \alpha_0^2 (G'/G)^2 + 4\alpha_0^3 \alpha_1 (G'/G) + \alpha_0^4\end{aligned}\quad (4.12)$$

Then by substituting (4.8) (4.9) (4.10) (4.11) and (4.12) into (4.7) , we find

$$\begin{aligned}& 45 \left(\alpha_1^4 (G'/G)^4 + 4\alpha_1^3 \alpha_0 (G'/G)^3 + \right. \\ & \left. 6\alpha_1^2 \alpha_0^2 (G'/G)^2 + 4\alpha_0^3 \alpha_1 (G'/G) + \alpha_0^4 \right) - \\ & 9\mathcal{W} \left(\alpha_1^2 (G'/G)^2 + 2\alpha_1 \alpha_0 (G'/G) + \alpha_0^2 \right) + \\ & 6 \left(2\alpha_1 (G'/G)^3 + 3\alpha_1 \lambda (G'/G)^2 + \right) \left((\alpha_0 + \alpha_1 (G'/G)) - \right. \\ & \left. (2\alpha_1 \mu + \alpha_1 \lambda^2) (G'/G) + \alpha_1 \lambda \mu \right) \\ & 2 \left(-\alpha_1 (G'/G)^2 - \alpha_1 \lambda (G'/G) - \alpha_1 \mu \right)^2 = 0.\end{aligned}\quad (4.13)$$

Or

$$\begin{aligned}& 45\alpha_1^4 (G'/G)^4 + 180\alpha_1^3 \alpha_0 (G'/G)^3 + 270\alpha_1^2 \alpha_0^2 (G'/G)^2 + \\ & 180\alpha_0^3 \alpha_1 (G'/G) + 45\alpha_0^4 - 9\mathcal{W} \alpha_1^2 (G'/G)^2 - \\ & 18\mathcal{W} \alpha_1 \alpha_0 (G'/G) - 9\mathcal{W} \alpha_0^2 + 12\alpha_0 \alpha_1 (G'/G)^3 + \\ & 18\alpha_0 \alpha_1 \lambda (G'/G)^2 + (12\alpha_0 \alpha_1 \mu + 6\alpha_0 \alpha_1 \lambda^2) (G'/G) + \\ & 6\alpha_0 \alpha_1 \lambda \mu + 12\alpha_1^2 (G'/G)^4 + 18\alpha_1^2 \lambda (G'/G)^3 + \\ & (12\alpha_1^2 \mu + 6\alpha_1^2 \lambda^2) (G'/G)^2 + 6\alpha_1^2 \lambda \mu (G'/G) - \\ & 2\alpha_1^2 (G'/G)^4 - 4\alpha_1^2 \lambda (G'/G)^3 - \\ & 2(2\alpha_1^2 \mu + \alpha_1^2 \lambda^2) (G'/G)^2 - 4\alpha_1^2 \lambda \mu (G'/G) - 2\alpha_1^2 \mu^2 = 0.\end{aligned}\quad (4.14)$$

Then, we collect the coefficients of (G'/G) as:

$$\begin{aligned}
& (45\alpha_1^4 + 10\alpha_1^2)(G'/G)^4 + (180\alpha_1^3\alpha_0 + 12\alpha_0\alpha_1 + 14\alpha_1^2\lambda)(G'/G)^3 + \\
& (270\alpha_1^2\alpha_0^2 - 9V\alpha_1^2 + 18\alpha_0\alpha_1\lambda + 8\alpha_1^2\mu + 4\alpha_1^2\lambda^2)(G'/G)^2 + \\
& (180\alpha_0^3\alpha_1 - 18V\alpha_1\alpha_0 + 12\alpha_0\alpha_1\mu + 6\alpha_0\alpha_1\lambda^2 + 2\alpha_1^2\lambda\mu)(G'/G) + \\
& 6\alpha_0\alpha_1\lambda\mu - 9V\alpha_0^2 + 45\alpha_0^4 - 2\alpha_1^2\mu^2 = 0.
\end{aligned} \tag{4.15}$$

This leads to the algebraic equations

$$\begin{aligned}
45\alpha_1^4 + 10\alpha_1^2 &= 0. \\
180\alpha_1^3\alpha_0 + 12\alpha_0\alpha_1 + 14\alpha_1^2\lambda &= 0. \\
270\alpha_1^2\alpha_0^2 - 9V\alpha_1^2 + 18\alpha_0\alpha_1\lambda + 8\alpha_1^2\mu + 4\alpha_1^2\lambda^2 &= 0. \\
180\alpha_0^3\alpha_1 - 18V\alpha_1\alpha_0 + 12\alpha_0\alpha_1\mu + 6\alpha_0\alpha_1\lambda^2 + 2\alpha_1^2\lambda\mu &= 0. \\
6\alpha_0\alpha_1\lambda\mu - 9V\alpha_0^2 + 45\alpha_0^4 - 2\alpha_1^2\mu^2 &= 0.
\end{aligned} \tag{4.16}$$

Next, we use the Maple program tools, to solve the algebraic equations (4.16) for α_0 , α_1 and V . This gives

$$V = 5\alpha_0^2, \quad \alpha_0 = \alpha_0, \quad \alpha_1 = 0, \tag{4.16}$$

Hence this solution is rejected because $\alpha_1 \neq 0 \in R$. In the Maple program listed below, we set $G'/G = \phi$.

```

> restart
> with(PDETools):
> declare(u(x, t), v(x, t), u(xi), phi(xi), prime=xi)
      u(x, t) will now be displayed as u
      v(x, t) will now be displayed as v
      u(xi) will now be displayed as u
      phi(xi) will now be displayed as phi
      derivatives with respect to xi of functions of one variable will now be displayed with '
(1)
> C - V u(xi) + 5 u^4(xi) + diff(u(xi), xi, xi) = 0
      C - V u + 5 u^4 + u'' = 0
(2)
> u(xi) = Phi^(2/3)(xi)
      u = Phi(xi)^2/3
(3)
> eval((2), (3))
      C - V Phi(xi)^2/3 + 5 Phi(xi)^8/3 - 2 Phi''^2 / (9 Phi(xi)^4/3) + 2 Phi' / (3 Phi(xi)^1/3) = 0
(4)
> simplify((4), 'symbolic')
      45 Phi(xi)^4 - 9 V Phi(xi)^2 + 9 C Phi(xi)^4/3 + 6 Phi'' Phi(xi) - 2 Phi'^2 = 0
(5)
> 45 Phi(xi)^4 - 9 V Phi(xi)^2 + 6 diff(diff(Phi(xi), xi), xi) Phi(xi) - 2 diff(Phi(xi), xi)^2 = 0
      45 Phi(xi)^4 - 9 V Phi(xi)^2 + 6 Phi'' Phi(xi) - 2 Phi'^2 = 0
(6)
> 4 n = n + n + 2
      4 n = 2 n + 2
(7)
> solve({(7)}, [n])
      [[n = 1]]
(8)
> Phi(xi) = alpha[0] + alpha[1] phi(xi)
      Phi(xi) = alpha_0 + alpha_1 phi
(9)
> diff(phi(xi), xi) = -lambda phi(xi) - mu - phi^2(xi)
      phi' = -lambda phi - mu - phi^2
(10)
> diff((9), xi)
      Phi' = alpha_1 phi'
(11)
> eval((11), (10))
      Phi' = alpha_1 (-lambda phi - mu - phi^2)
(12)
> diff((12), xi)
      Phi'' = alpha_1 (-lambda phi' - 2 phi phi')
(13)
> eval((13), (10))
      Phi'' = alpha_1 (-lambda (-lambda phi - mu - phi^2) - 2 phi (-lambda phi - mu - phi^2))
(14)
> simplify((14), 'symbolic')
      Phi'' = alpha_1 (phi^2 + lambda phi + mu) (lambda + 2 phi)
(15)
> eval((6), {(9), (12), (15)})
      45 (alpha_0 + alpha_1 phi)^4 - 9 V (alpha_0 + alpha_1 phi)^2 + 6 alpha_1 (phi^2 + lambda phi + mu) (lambda + 2 phi) (alpha_0 + alpha_1 phi) - 2 alpha_1^2 (-lambda phi - mu - phi^2)^2 = 0
(16)
> expand((16))
      45 alpha_0^4 + 180 alpha_0^3 alpha_1 phi + 270 alpha_0^2 alpha_1^2 phi^2 + 180 alpha_0 alpha_1^3 phi^3 + 45 alpha_1^4 phi^4 - 9 V alpha_0^2 - 18 V alpha_0 alpha_1 phi - 9 V alpha_1^2 phi^2 + 18 alpha_1 phi^2 lambda alpha_0 + 14 phi^3 lambda alpha_1^2
      + 12 alpha_1 phi^3 alpha_0 + 10 phi^4 alpha_1^2 + 6 alpha_1 lambda^2 phi alpha_0 + 4 lambda^2 phi^2 alpha_1^2 + 6 alpha_1 mu lambda alpha_0 + 2 mu lambda alpha_1^2 phi + 12 alpha_1 mu phi alpha_0 + 8 mu phi^2 alpha_1^2 - 2 alpha_1^2 mu^2 = 0
(17)

```

```

> collect((17), phi(xi))
(45 alpha_1^4 + 10 alpha_1^2) phi^4 + (180 alpha_0 alpha_1^3 + 14 lambda alpha_1^2 + 12 alpha_0 alpha_1) phi^3 + (4 lambda^2 alpha_1^2 + 270 alpha_0^2 alpha_1^2 - 9 V alpha_1^2 + 18 lambda alpha_0 alpha_1 + 8 mu alpha_1^2) phi^2 + (6 lambda^2 alpha_0 alpha_1
+ 2 lambda mu alpha_1^2 + 180 alpha_0^3 alpha_1 - 18 V alpha_0 alpha_1 + 12 mu alpha_0 alpha_1) phi + 6 alpha_1 mu lambda alpha_0 - 2 alpha_1^2 mu^2 + 45 alpha_0^4 - 9 V alpha_0^2 = 0 (18)
> {(45 alpha_1^4 + 10 alpha_1^2) = 0, (180 alpha_0 alpha_1^3 + 14 lambda alpha_1^2 + 12 alpha_0 alpha_1) = 0, (4 lambda^2 alpha_1^2 + 270 alpha_0^2 alpha_1^2 - 9 V alpha_1^2 + 18 lambda alpha_0 alpha_1 + 8 mu alpha_1^2) = 0, (6 lambda^2 alpha_0 alpha_1
+ 2 lambda mu alpha_1^2 + 180 alpha_0^3 alpha_1 - 18 V alpha_0 alpha_1 + 12 mu alpha_0 alpha_1) = 0, (6 alpha_1 mu lambda alpha_0 - 2 alpha_1^2 mu^2 + 45 alpha_0^4 - 9 V alpha_0^2) = 0}
{45 alpha_1^4 + 10 alpha_1^2 = 0, 180 alpha_0 alpha_1^3 + 14 lambda alpha_1^2 + 12 alpha_0 alpha_1 = 0, 6 alpha_1 mu lambda alpha_0 - 2 alpha_1^2 mu^2 + 45 alpha_0^4 - 9 V alpha_0^2 = 0, 4 lambda^2 alpha_1^2 + 270 alpha_0^2 alpha_1^2 - 9 V alpha_1^2
+ 18 lambda alpha_0 alpha_1 + 8 mu alpha_1^2 = 0, 6 lambda^2 alpha_0 alpha_1 + 2 lambda mu alpha_1^2 + 180 alpha_0^3 alpha_1 - 18 V alpha_0 alpha_1 + 12 mu alpha_0 alpha_1 = 0} (19)
> solve((19), {alpha[0], alpha[1], V})
{V = V, alpha_0 = 0, alpha_1 = 0}, {V = 5 alpha_0^2, alpha_0 = alpha_0, alpha_1 = 0} (20)

```

Maple Programme for $u(\xi) = \Phi^{\frac{2}{3}}(\xi)$

(4.2) A New Approach

In the cases, when the homogeneous balance number equal to m , is not a positive integer number, (Zayed & Alurrfi, 2014), suggest the transform (1.6) to reduce the homogeneous balance number to positive integer. In the previous section, we found $m = 2/3$, then transform (1.6) reduced the equation (4.1) to an equation with homogeneous balance number $m = 1$. For the resulting equation, the method led to a rejected solution (4.17). As a suggested approach, we introduce the transform $u(\xi) = \Phi^{N-m}(\xi)$, where N is the smallest positive integer greater than m . Using this transform, we were able to reduce the equation (4.1) to an equation with a positive integer homogeneous balance number.

For the equation (4.3), we have $m = 2/3$, $N = 1$. Hence, we can apply the transform

$$u(\xi) = \Phi^{\frac{1}{3}}(\xi) \quad (4.18)$$

Then (4.5) be

$$C - V(\Phi^{\frac{1}{3}}) + 5\Phi^{\frac{4}{3}} - \frac{2(\Phi')^2}{9\Phi^{\frac{5}{3}}} + \frac{(\Phi'')}{3\Phi^{\frac{2}{3}}} = 0, \quad (4.19)$$

Or

$$5\Phi^3 - V(\Phi^2) + C(\Phi^{\frac{5}{3}}) + \frac{1}{3}(\Phi'')(\Phi) - \frac{2}{9}(\Phi')^2 = 0. \quad (4.20)$$

Hence (4.20) is integrable if $C = 0$, this gives

$$5\Phi^3 - V(\Phi^2) + \frac{1}{3}(\Phi'')(\Phi) - \frac{2}{9}(\Phi')^2 = 0. \quad (4.21)$$

$$45\Phi^3 - 9V(\Phi^2) + 3(\Phi'')(\Phi) - 2(\Phi')^2 = 0. \quad (4.22)$$

By balancing (4.22), then we find ($3m = m + 2 + m \Rightarrow m = 2$), so

$$\Phi(\xi) = \sum_{i=0}^2 \alpha_i (G'/G)^i = \alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2, \quad \alpha_2 \neq 0 \quad (4.23)$$

$$\Phi^2(\xi) = \alpha_2^2 (G'/G)^4 + 2\alpha_1\alpha_2 (G'/G)^3 + (\alpha_1^2 + 2\alpha_1\alpha_2)(G'/G)^2 + 2\alpha_1\alpha_0 (G'/G) + \alpha_0^2 \quad (4.24)$$

$$\begin{aligned} \Phi^3(\xi) = & \alpha_2^3 (G'/G)^6 + 3\alpha_2^2\alpha_1 (G'/G)^5 + \\ & (3\alpha_2^2\alpha_0 + 3\alpha_1^2\alpha_2)(G'/G)^4 + (\alpha_1^3 + 6\alpha_0\alpha_1\alpha_2)(G'/G)^3 + \\ & (3\alpha_1^2\alpha_0 + 3\alpha_0^2\alpha_2)(G'/G)^2 + 3\alpha_0^2\alpha_1 (G'/G) + \alpha_0^3 \end{aligned} \quad (4.25)$$

$$\begin{aligned} \Phi' = & -2\alpha_2 (G'/G)^3 - (2\alpha_2\lambda + \alpha_1)(G'/G)^2 - \\ & (\alpha_1\lambda + 2\alpha_2\mu)(G'/G) - \alpha_1\mu \end{aligned} \quad (4.26)$$

$$\begin{aligned} \Phi'' = & 6\alpha_2 (G'/G)^4 + (10\alpha_2\lambda + 2\alpha_1)(G'/G)^3 + (8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(G'/G)^2 \\ & + (6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)(G'/G) + \alpha_1\lambda\mu + 2\alpha_2\mu^2 \end{aligned} \quad (4.27)$$

Then (4.22) be

$$\begin{aligned}
& 45(\alpha_2^3 (G'/G)^6 + 3\alpha_2^2 \alpha_1 (G'/G)^5 + (3\alpha_2^2 \alpha_0 + 3\alpha_1^2 \alpha_2)(G'/G)^4 + \\
& (\alpha_1^3 + 6\alpha_0 \alpha_1 \alpha_2)(G'/G)^3 + (3\alpha_1^2 \alpha_0 + 3\alpha_0^2 \alpha_2)(G'/G)^2 + 3\alpha_0^2 \alpha_1 (G'/G) \\
& + \alpha_0^3) - 9\mathcal{V} (\alpha_2^2 (G'/G)^4 + 2\alpha_1 \alpha_2 (G'/G)^3 + (\alpha_1^2 + 2\alpha_1 \alpha_2)(G'/G)^2 \\
& + 2\alpha_1 \alpha_2 (G'/G) + \alpha_0^2) + 3(6\alpha_2 (G'/G)^4 + (10\alpha_2 \lambda + 2\alpha_1)(G'/G)^3 \\
& + (8\alpha_2 \mu + 3\alpha_1 \lambda + 4\alpha_2 \lambda^2)(G'/G)^2 + (6\alpha_2 \lambda \mu + 2\alpha_1 \mu + \alpha_1 \lambda^2)(G'/G) \\
& + \alpha_1 \lambda \mu + 2\alpha_2 \mu^2)(\alpha_0 + \alpha_1 (G'/G) + \alpha_2 (G'/G)^2) - 2(-2\alpha_2 (G'/G)^3 \\
& - (2\alpha_2 \lambda + \alpha_1)(G'/G)^2 - (\alpha_1 \lambda + 2\alpha_2 \mu)(G'/G) - \alpha_1 \mu)^2 = 0.
\end{aligned} \tag{4.28}$$

Next, we expand (4.28) and collect the coefficients of (G'/G) . This gives the algebraic equations

$$\begin{aligned}
& (45\alpha_2^3 + 10\alpha_2^2) = 0 \\
& (135\alpha_2^2 \alpha_1 + 14\alpha_2^2 \lambda + 16\alpha_1 \alpha_2) = 0 \\
& \left(\begin{array}{l} 135\alpha_2^2 \alpha_0 + 135\alpha_1^2 \alpha_2 - 9\mathcal{V} \alpha_2^2 + 8\alpha_2^2 \mu + \\ 23\alpha_1 \alpha_2 \lambda + 4\alpha_2^2 \lambda^2 + 4\alpha_1^2 + 18\alpha_2 \alpha_0 \end{array} \right) = 0 \\
& \left(\begin{array}{l} 45\alpha_1^3 + 270\alpha_0 \alpha_1 \alpha_2 - 18\mathcal{V} \alpha_1 \alpha_2 + 2\alpha_2^2 \lambda \mu + \\ 14\alpha_1 \alpha_2 \mu + 5\alpha_1^2 \lambda + 30\alpha_2 \alpha_0 \lambda + 6\alpha_1 \alpha_0 + 7\alpha_1 \alpha_2 \lambda^2 \end{array} \right) = 0 \\
& \left(\begin{array}{l} 135\alpha_1^2 \alpha_0 + 135\alpha_0^2 \alpha_2 - 9\mathcal{V} \alpha_1^2 - 18\mathcal{V} \alpha_0 \alpha_2 \\ + 5\alpha_2 \alpha_1 \mu \lambda - 2\alpha_2^2 \mu^2 + 2\alpha_1^2 \mu + \alpha_1^2 \lambda^2 \\ + 24\alpha_2 \alpha_0 \mu + 9\alpha_1 \alpha_0 \lambda + 12\alpha_2 \alpha_0 \lambda^2 \end{array} \right) = 0 \\
& \left(\begin{array}{l} 135\alpha_0^2 \alpha_1 - 18\mathcal{V} \alpha_1 \alpha_0 - \alpha_1^2 \lambda \mu - 2\alpha_2 \alpha_1 \mu^2 + \\ 18\alpha_2 \alpha_0 \lambda \mu + 6\alpha_1 \alpha_0 \mu + 3\alpha_1 \alpha_0 \lambda^2 \end{array} \right) = 0 \\
& (45\alpha_0^3 - 9\mathcal{V} \alpha_0^2 + 3\alpha_1 \alpha_0 \lambda \mu + 6\alpha_2 \alpha_0 \mu^2 - 2\alpha_1^2 \mu^2) = 0.
\end{aligned} \tag{4.29}$$

Next, by using the Maple program tools listed below, the solution of the algebraic equations (4.29) for α_0 , α_1 , α_2 and V reads

$$V = \frac{\lambda^2}{9} - \frac{4\mu}{9}, \quad \alpha_0 = \frac{-2\mu}{9}, \quad \alpha_1 = \frac{-2\lambda}{9}, \quad \alpha_2 = \frac{-2}{9}. \tag{4.30}$$

```

> restart
> with(PDETools):
> declare(u(x, t), v(x, t), u(xi), phi(xi), prime = xi)
      u(x, t) will now be displayed as u
      v(x, t) will now be displayed as v
      u(xi) will now be displayed as u
      phi(xi) will now be displayed as phi
      derivatives with respect to xi of functions of one variable will now be displayed with '
(1)
> C - V u(xi) + 5 u^4(xi) + diff(u(xi), xi, xi) = 0
      C - V u + 5 u^4 + u'' = 0
(2)
> u(xi) = Phi^(1/3)(xi)
      u = Phi(xi)^1/3
(3)
> eval((2), (3))
      C - V Phi(xi)^1/3 + 5 Phi(xi)^4/3 - 2 Phi^2 / (9 Phi(xi)^5/3) + Phi' / (3 Phi(xi)^2/3) = 0
(4)
> simplify((4), 'symbolic')
      9 C Phi(xi)^5/3 + 3 Phi' Phi(xi) - 2 Phi^2 - 9 Phi(xi)^2 (V - 5 Phi(xi)) = 0
(5)
> 3 diff(diff(Phi(xi), xi), xi) Phi(xi) - 2 diff(Phi(xi), xi)^2 - 9 Phi(xi)^2 (V - 5 Phi(xi)) = 0
      3 Phi' Phi(xi) - 2 Phi^2 - 9 Phi(xi)^2 (V - 5 Phi(xi)) = 0
(6)
> 3 n = n + n + 2
      3 n = 2 n + 2
(7)
> solve({(7)}, [n])
      [[n=2]]
(8)
...
> Phi(xi) = alpha[0] + alpha[1] phi(xi) + alpha[2] phi(xi)^2
      Phi(xi) = alpha_0 + phi alpha_1 + phi^2 alpha_2
(9)
> diff(phi(xi), xi) = -lambda phi(xi) - mu - phi^2(xi)
      phi' = -lambda phi - mu - phi^2
(10)
> diff((9), xi)
      Phi' = phi' alpha_1 + 2 phi alpha_2 phi'
(11)
> eval((11), (10))
      Phi' = alpha_1 (-lambda phi - mu - phi^2) + 2 alpha_2 phi (-lambda phi - mu - phi^2)
(12)
> diff((12), xi)
      Phi'' = alpha_1 (-lambda phi' - 2 phi phi') + 2 alpha_2 phi' (-lambda phi - mu - phi^2) + 2 alpha_2 phi (-lambda phi' - 2 phi phi')
(13)
> eval((13), (10))
      Phi'' = alpha_1 (-lambda (-lambda phi - mu - phi^2) - 2 phi (-lambda phi - mu - phi^2)) + 2 alpha_2 (-lambda phi - mu - phi^2)^2 + 2 alpha_2 phi (-lambda (-lambda phi - mu - phi^2) - 2 phi (-lambda phi - mu - phi^2))
(14)
> simplify((14), 'symbolic')
      Phi'' = 2 (3 phi^2 alpha_2 + (2 lambda alpha_2 + alpha_1) phi + mu alpha_2 + lambda alpha_1 / 2) (lambda phi + phi^2 + mu)
(15)

```

```

> eval((6), {(9), (12), (15)})
6 (3 ρ² α₂ + (2 λ α₂ + α₁) ρ + μ α₂ +  $\frac{\lambda \alpha_1}{2}$ ) (λ ρ + ρ² + μ) (α₀ + ρ α₁ + ρ² α₂) - 2 (α₁ (-λ ρ - μ - ρ²) + 2 α₂ ρ (-λ ρ - μ - ρ²))² - 9 (α₀ + ρ α₁ + ρ² α₂)² (V - 5 α₀ - 5 ρ α₁ - 5 ρ² α₂) = 0 (16)
> expand(16)
3 λ α₁ μ α₀ + 270 α₀ ρ³ α₁ α₂ - 18 V α₀ ρ α₁ - 18 V α₀ ρ² α₂ - 18 V ρ³ α₁ α₂ + 30 ρ³ α₂ λ α₀ + 23 ρ⁴ α₂ λ α₁ + 24 ρ² α₂ μ α₀ + 14 ρ³ α₂ μ α₁ (17)
+ 12 ρ² λ² α₂ α₀ + 7 ρ³ λ² α₂ α₁ + 2 ρ³ λ α₂² μ + 9 ρ² α₁ λ α₀ + 6 ρ α₁ μ α₀ - 2 μ² α₂ ρ α₁ - 9 V α₀² - 2 α₁² μ² - λ α₁² μ ρ + 45 α₀³
+ 6 μ² α₂ α₀ + 18 ρ λ α₂ μ α₀ + 5 ρ² λ α₂ μ α₁ + 3 λ² α₁ ρ α₀ + 45 ρ³ α₁³ + 45 ρ⁶ α₂³ + 10 ρ⁶ α₂² + 4 ρ⁴ α₁² + 14 ρ⁵ α₂² λ + 18 ρ⁴ α₂ α₀
+ 16 ρ⁵ α₂ α₁ + 8 ρ⁴ α₂² μ + 4 ρ⁴ λ² α₂² + 5 ρ³ α₁² λ + 6 ρ³ α₁ α₀ + 2 ρ² α₁² μ - 2 μ² α₂² ρ² + λ² α₁² ρ² - 9 V ρ² α₁² - 9 V ρ⁴ α₂² + 135 α₀ ρ²
α₁² + 135 α₀ ρ⁴ α₂² + 135 ρ⁴ α₁² α₂ + 135 ρ⁵ α₁ α₂² + 135 α₀² ρ α₁ + 135 α₀² ρ² α₂ = 0
> collect(17), ρ(ξ)
(45 α₂³ + 10 α₂²) ρ⁶ + (14 λ α₂² + 135 α₁ α₂² + 16 α₁ α₂) ρ⁵ + (4 λ² α₂² - 9 V α₂² + 23 λ α₁ α₂ + 8 μ α₂² + 135 α₀ α₂² + 135 α₁² α₂ (18)
+ 18 α₀ α₂ + 4 α₁²) ρ⁴ + (7 λ² α₁ α₂ + 2 λ μ α₂² - 18 V α₁ α₂ + 30 λ α₀ α₂ + 5 λ α₁² + 14 μ α₁ α₂ + 270 α₀ α₁ α₂ + 45 α₁³
+ 6 α₀ α₁) ρ³ + (12 λ² α₀ α₂ + λ² α₁² + 5 λ μ α₁ α₂ - 2 μ² α₂² - 18 V α₀ α₂ - 9 V α₁² + 9 λ α₀ α₁ + 24 μ α₀ α₂ + 2 μ α₁² + 135 α₀² α₂
+ 135 α₀ α₁²) ρ² + (3 λ² α₀ α₁ + 18 λ μ α₀ α₂ - λ μ α₁² - 2 μ² α₁ α₂ - 18 V α₀ α₁ + 6 μ α₀ α₁ + 135 α₀² α₁) ρ + 3 λ α₁ μ α₀
+ 6 μ² α₂ α₀ - 2 α₁² μ² - 9 V α₀² + 45 α₀³ = 0
> {(45 α₂³ + 10 α₂²) = 0, (14 λ α₂² + 135 α₁ α₂² + 16 α₁ α₂) = 0, (4 λ² α₂² - 9 V α₂² + 23 λ α₁ α₂ + 8 μ α₂² + 135 α₀ α₂² + 135 α₁² α₂
+ 18 α₀ α₂ + 4 α₁²) = 0, (7 λ² α₁ α₂ + 2 λ μ α₂² - 18 V α₁ α₂ + 30 λ α₀ α₂ + 5 λ α₁² + 14 μ α₁ α₂ + 270 α₀ α₁ α₂ + 45 α₁³ + 6 α₀ α₁)
= 0, (12 λ² α₀ α₂ + λ² α₁² + 5 λ μ α₁ α₂ - 2 μ² α₂² - 18 V α₀ α₂ - 9 V α₁² + 9 λ α₀ α₁ + 24 μ α₀ α₂ + 2 μ α₁² + 135 α₀² α₂ + 135 α₀
α₁²) = 0, (3 λ² α₀ α₁ + 18 λ μ α₀ α₂ - λ μ α₁² - 2 μ² α₁ α₂ - 18 V α₀ α₁ + 6 μ α₀ α₁ + 135 α₀² α₁) = 0, (3 λ α₁ μ α₀ + 6 μ² α₂ α₀ - 2
α₁² μ² - 9 V α₀² + 45 α₀³) = 0}
{45 α₂³ + 10 α₂² = 0, 14 λ α₂² + 135 α₁ α₂² + 16 α₁ α₂ = 0, 3 λ α₁ μ α₀ + 6 μ² α₂ α₀ - 2 α₁² μ² - 9 V α₀² + 45 α₀³ = 0, 3 λ² α₀ α₁ + 18 λ μ α₀ α₂ (19)
- λ μ α₁² - 2 μ² α₁ α₂ - 18 V α₀ α₁ + 6 μ α₀ α₁ + 135 α₀² α₁ = 0, 4 λ² α₂² - 9 V α₂² + 23 λ α₁ α₂ + 8 μ α₂² + 135 α₀ α₂² + 135 α₁² α₂
+ 18 α₀ α₂ + 4 α₁² = 0, 7 λ² α₁ α₂ + 2 λ μ α₂² - 18 V α₁ α₂ + 30 λ α₀ α₂ + 5 λ α₁² + 14 μ α₁ α₂ + 270 α₀ α₁ α₂ + 45 α₁³ + 6 α₀ α₁ = 0,
12 λ² α₀ α₂ + λ² α₁² + 5 λ μ α₁ α₂ - 2 μ² α₂² - 18 V α₀ α₂ - 9 V α₁² + 9 λ α₀ α₁ + 24 μ α₀ α₂ + 2 μ α₁² + 135 α₀² α₂ + 135 α₀ α₁² = 0}
> solve(19), {α[0], α[1], α[2], V}
{V = V, α₀ = 0, α₁ = 0, α₂ = 0}, {V = 5 α₀, α₀ = α₀, α₁ = 0, α₂ = 0}, {V =  $\frac{\lambda^2}{9} - \frac{4\mu}{9}$ , α₀ =  $-\frac{2\mu}{9}$ , α₁ =  $-\frac{2\lambda}{9}$ , α₂ =  $-\frac{2}{9}$ } (20)
>
> {V =  $\frac{\lambda^2}{9} - \frac{4\mu}{9}$ , α₀ =  $-\frac{2\mu}{9}$ , α₁ =  $-\frac{2\lambda}{9}$ , α₂ =  $-\frac{2}{9}$ } (21)
{V =  $\frac{\lambda^2}{9} - \frac{4\mu}{9}$ , α₀ =  $-\frac{2\mu}{9}$ , α₁ =  $-\frac{2\lambda}{9}$ , α₂ =  $-\frac{2}{9}$ }

```

Maple Programme for $u(\xi) = \Phi^{\frac{1}{3}}(\xi)$

Conclusion

In this work, we reviewed the (G'/G) -expansion method, to find exact wave solutions for nonlinear partial differential evolution equations. And as applications, with the help of the Maple program tools, we solve the Burger's equation, the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation, the general KdV equation for $p = 1$, the Boussinesq's equation and the variant Boussinesq's equation, which have positive integer homogeneous balance numbers.

In the cases when the homogeneous balance number is not positive integer, we suggested a new approach to reduce the homogeneous balance number to a positive integer. As an application, we applied this approach successfully to find a travelling wave solution to the general KdV equation for $p = 3$. We believe that a further research work must be carry out to study this approach for the non-linear evolution equations, which involving non positive integer homogeneous balance numbers.

As a comparison study, we compare our results with results obtained by other methods. This study shows that the (G'/G) -expansion method is direct, concise, elementary and effective, and can be used to solve many nonlinear evolution equations with constant coefficients.

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الخلاصة

في هذا البحث، استعرضنا طريقة التوسيع (G'/G) لإيجاد حلول موجية دقيقة لمعادلات التطور التفاضلي الجزئي غير الخطية. وكتطبيقات وبمساعدة أدوات برنامج Maple ، حللنا معادلة برغر، ومعادلة زاخاروف-كوزنيثسوف-بنيامين-بونا-ماهوني (ZK-BBM) ، والشكل العام لمعادلة KdV لـ $p = 1$ ، ومعادلة بوسينسك، ومعادلة بوسينسك المتغيرة، والتي تحتوي على أعداد توازن متجانسة صحيحة موجبة.

في الحالات التي لا يكون فيها عدد التوازن المتجانس عددًا صحيحًا، اقترحنا نهجًا جديدًا لاختزل عدد التوازن المتجانس إلى عدد صحيح موجب، وكتطبيق عملي، طبقنا هذا النهج بنجاح لإيجاد حل موجة متنقلة لمعادلة KdV العامة لـ $p = 3$. ونعتقد أنه يجب إجراء المزيد من الأبحاث لدراسة هذا النهج على معادلات التطور غير الخطية، التي تتضمن أعدادًا متوازنة متجانسة صحيحة غير موجبة. وهذه ستكون جزء من دراستنا في المستقبل بأذن الله تعالى.

كدراسة مقارنة، قارنا نتائجنا بالنتائج التي تم الحصول عليها بطرق أخرى، وتبين هذه الدراسة أن طريقة التوسيع (G'/G) مباشرة، موجزة، بسيطة، وفعالة، ويمكن استخدامها لحل العديد من معادلات التطور غير الخطية ذات المعاملات الثابتة.



جامعة بنغازي
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