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THE BEURLING AND CAUCHY TRANSFORM WITH NEW SPACES

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Abstract. Let X be a compact plane set. Then $R(X)$ is the uniform algebra of all continuous functions on X which may be uniformly approximated on X by rational functions with poles off X. Some results which contain the Beuring and Cauchy transforms are true for $R(X)$. Our aim in this paper is to change the uniform algebra $R(X)$ with some other new uniform algebras, especially $A(X)$ which is the uniform algebra of all holomorphic functions on the interior of X.

Of course, the proof of these results are completely different and new after this change. We have proved three results in this area which are still true after changing $R(X)$ by $A(X)$.

1. Introduction. In this section, we shall give some standard definitions which we shall need in this paper. We shall also give some notation and related remarks.

Throughout, we shall consider only linear spaces and algebras over the complex field $\mathbb{C}$. If an algebra has an identity, we shall denote this identity by 1.

Let S be a non-empty set, and let f be a bounded, complex-valued function on S. For each non-empty set E contained in S, the uniform norm of f on E, denoted by $|f|_E$, is defined by

$$|f|_E = \sup\{|f(x)| : x \in E\}.$$

A normed algebra is an algebra A, equipped with a norm $\|\cdot\|$ which is submultiplicative. That is

$$\|ab\| \leq \|a\| \, \|b\| \quad (a, b \in A)$$

If A is unital, then we also require that $\|1\| = 1$.

A Banach algebra is a normed algebra which is completed as a normed space.

In our terminology a compact space is a compact Hausdorff topological space.

Notation. Let X be a compact space. We shall denoted by $C(X)$ the algebra of all continuous functions from X into $\mathbb{C}$. With respect to the uniform norm on X, $C(X)$ becomes a Banach algebra.

Let X be a compact space, and let $A \subseteq C(X)$. Then A separates the points of X if for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$.

Let A be a complex algebra. A left A-module (respectively right A-module) is a linear space X together with a bilinear map $(a, x) \mapsto a \cdot x$ (respectively $(a, x) \mapsto x \cdot a$) from $A \times X$ into X satisfying

$$(ab) \cdot x = a \cdot (b \cdot x) \text{ (respectively } x \cdot (ab) = (x \cdot a) \cdot b) \quad (a, b \in A, \ x \in X).$$

An A-bimodule is a left A-module X which is also a right A-module, and satisfies

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b \quad (a, b \in A \ x \in X).$$

The A-bimodule X is commutative if $a \cdot x = x \cdot a$ $(a \in A, \ x \in X)$. If A is commutative, then an A- module is a commutative A-bimodule.

Note that an algebra A is always an A-bimodule with module operations given by multiplication in A.

Let A be a Banach algebra. A derivation from A into an A-bimodule X is a linear map $D : A \to X$ satisfying

$$D(ab) = a \cdot Db + D \cdot b \quad (a, b \in A).$$

A continuous derivation is derivation which is also continuous.

The polynomially convex hull of a compact subset X of $\mathbb{C}^n$, denoted by $\hat{X}$, consists of all $z \in \mathbb{C}^n$ such that

$$|p(z)| \leq \sup_{w \in X} |p(w)|,$$

for all polynomials p.

It is standard that $\hat{X}$, is a compact subset of $\mathbb{C}^n$ containing X (see, for example, (Gamelin 1969).

Let X be a compact space. A uniform algebra on X is a subalgebra of $C(X)$ which contains all the constant functions and separates the points of X with respect to the uniform norm on X.

Note that every uniform algebra is a closed subalgebra of $C(X)$.

We now give important examples of a family of uniform algebras.

Let X be a compact subset of $\mathbb{C}$. Then $R_0(X)$ is the set of restrictions to X of rational functions with poles off X, and $R(X)$ is the closure of $R_0(X)$ in $C(X)$. (That is $\overline{R_0(X)} = R(X)$).

Let X be compact subset of $|C$. Then $A(X)$ is the set of all holomorphic functions on the interior of X.

REMARK. Let X be a compact subset of $\mathbb{C}$. it is standard that $A(X) = C(X)$ if and only if $\text{int}\,(X) = \emptyset$.

Let X be a compact subset of $\mathbb{C}$. Then $P_0(X)$ is the set of restrictions to X of polynomials with complex coefficients and $P(X)$ is the closure of $P_0(X)$ in $C(X)$. (That is $\overline{P_0(X)} = P(X)$.).

Let X be a compact space and let $A \subset C(X)$. Let μ be a measure on X. Then we say that μ annihilates A, and we write $\mu \perp A$, if

$$\int_X f\, d\mu = 0 \quad (f \in A).$$

Note that if X is a compact plane set and μ is a complex measure on X, then it is easy to see that $\mu \perp R_0(X)$ if and only if $\mu \perp R(X)$.

Similarly, $\mu \perp P_0(X)$ if and only if $\mu \perp P(X)$.

2. Some standard spaces. In this section, we introduce some standard spaces which are needed in the next section. Let us start with the following definition.

DEFINITION 2.1 *Let E be a linear space. A function $\|\cdot\|$ on E is a quasi-norm if it satisfies the following conditions :*

(i) $\|x\| \geq 0 \quad (x \in E)$;

(ii) $\|x\| = 0$ *if and only if* $x = 0$;

(iii) $\|\alpha x\| \|\alpha\| \|x\| \quad (x \in E, \alpha \in \mathbb{C})$;

(iv) *there exists* $k \geq 1$ *with* $\|x + y\| \leq k(\|x\| + \|y\|)$ $(x \cdot y \in E)$.

We call $(E, \|\cdot\|)$ a quasi-normed space.

Note that every quasi-normed space $(E, \|\cdot\|)$ is a topological vector space with respect to a suitable topology on E (see Kothe 1069).

REMARK. There are many discontinuous quasi-norms on the topological linear space $\mathbb{C}$. Thus it is not always true that a quasi-norm is continuous with respect to the topology that it generates. For example, we may take

$$\|z\| = \begin{cases} |z| & (z \in \mathbb{C} \backslash \mathbb{R}^+), \\ 2|z| & (z \in \mathbb{R}^+). \end{cases}$$

Let i be a complex number. We define a sequence z_n by

$$z_n = 1 + \frac{1}{n} i \quad (n \in \mathbb{N})$$
$$\text{and} \quad z = 1.$$

Then $z_n \to z$ in $\mathbb{C}$, but $\|z_n\| \not\to \|z\|$.

NOTATION. Let X be a locally compact space. Then we denote by $\mathcal{M}(X)$ the Banach space of all complex, regular Borel measures on X with the norm

$$\|\mu\| = |\mu|(X).$$

For $\mu \in \mathcal{M}(X)$, $|\mu|$ denotes the corresponding total variation measure.

DEFINITION 2.2 *Let X be a locally compact space, and let μ be a positive measure on X. Then weak-L^1 with respect to μ, denoted by $L^1_*(X, d\mu)$ is the set of all equivalence classes of Borel measurable functions f on X satisfying*

$$\sup_{t>0} t\mu(\{x \in X : |f(x)| > t\|) < \infty,$$

where the equivalence relation is almost everywhere equality with respect to μ. We define a function $\|\cdot\|$ on $L^1_*(X, d\mu)$ by

$$\|f\| = \sup_{t>0} t\mu(\{x \in X : |f(x)| > t\}) \quad (f \in L^1_*(X, d\mu)).$$

NOTATION. Let X be a locally compact space, and let μ be a positive, regular Borel measure on X. For $p \in (0, \infty)$, $L^p(X, d\mu)$ denotes the set of equivalence classes of all Borel measurable, complex-valued functions on X satisfying

$$\int_X |f(x)|^p d\mu(x) < \infty.$$

If $p \geq 1$, then $L^p(X, d\mu)$ is a Banach space with respect the norm $\|\cdot\|_p$, given by

$$\|f\|_p \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \quad (f \in L^p(X, d\mu)).$$

For each $p \in (0, 1)$, $L^p(X, d\mu)$ is a quasi-normed space. For more details concerning $L^p(X, d\mu)$, see, for example,(Feinstein 1989).

The following lemmas are a selection of standard facts about $L^1_*(X, d\mu)$. For the proofs, see (Feinstein 1989). In these lemmas, we assume that X is a fixed compact space, and μ a positive, regular Borel measure on X.

LEMMA 2.3 (i) *If $f, g \in L^1_*(X, d\mu)$, and $\lambda \in (0, 1)$, then $f + g \in L^1_*(X, d\mu)$, and*

$$\|f + g\| \leq (\lambda^{-1}\|f\| + (1 - \lambda)^{-1}\|g\|).$$

In particular, $\|f + g\| \leq 2(\|f\| + \|g\|)$.

(ii) *If $f \in L^1_*(X, d\mu)$ and $\alpha \in \mathbb{C}$, then $\alpha f \in L^1_*(X, d\mu)$ and $\|\alpha f\| = |\alpha|\|f\|$.*

(iii) $(L^1_*(X, d\mu), \|\cdot\|)$ *is a quasi-normed space.*

(iv) *The quasi-norm* $\|\cdot\|$ *is continuous on* $(L^1_*(X, d\mu), \|\cdot\|)$.

LEMMA 2.4 *If* μ *is a* σ*-finite, then* $(L^1_*(X, d\mu), \|\cdot\|)$ *is a complete space.*

LEMMA 2.5 *If* μ *is finite and* $p \in (0,1)$, *then* $L^1_*(X, d\mu)$ *is contained in* $L^p(X, d\mu)$, *and the inclusion map is continuous.*

LEMMA 2.6 *Let* g *be a measurable function on* X *and suppose that* (f_n) *is a sequence of elements of* $L^1_*(X, d\mu)$ *with* $f_n \to g$ *a.e.* (μ) *and with*

$$\sup\{\|f_n\| : n \in \mathbf{N}\} < \infty.$$

Then $g \in L^1_*(X, d\mu)$ *and* $\|f_n - g\| \leq \liminf_{m\to\infty} \|f_n - f_m\|$ $(n \in \mathbf{N})$.

3. The Beurling and Cauchy transforms with new spaces. Our work in this section is to change the uniform algebra $R(X)$ by the uniform algebra $A(X)$ in some results in the area of the Beurling and Cauchy transforms. We have proved some results are still true after this change. Some modifications in the statements of these results have been made.

We start with the following standard definitions.

Let $\mu \in \mathcal{M}(\mathbf{C})$. We get

$$\tilde{\mu}(z) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{d|\mu|(w)}{|w-z|} \quad (z \in \mathbf{C}).$$

where the integrand is defined as ∞ when $w = z$. For those $z \in \mathbf{C}$ with $\tilde{\mu}(z) < \infty$, we get

$$\hat{\mu}(z) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{d\mu(w)}{w-z} \quad (z \in \mathbf{C}).$$

Note that for all such z we have $\mu(\{z\}) = 0$. Clearly $|\hat{\mu}| \leq \tilde{\mu}$.

For the rest of this section, the Lebesgue measure on the plane is denoted by m. A measurable complex-valued function f on $\mathbf{C}$ is called locally integrable if for every compact set $E \subseteq \mathbf{C}$,

$$\int_E |f| dm < \infty.$$

We denote by $L^1_{loc}(\mathbf{C}, dm)$ the set of locally integrable, measurable, complex-valued functions on $\mathbf{C}$. It follows from Fubini's theorem that $\tilde{\mu} \in L^1_{loc}(\mathbf{C}, dm)$, and so $\hat{\mu} \in L^1_{loc}(\mathbf{C}, dm)$. The function $\hat{\mu}$ is called the Cauchy transform of μ.

For each $\varepsilon > 0$, we get

$$(B_\varepsilon \mu)(z) = \frac{1}{\pi}\int_{|w-z|\geq\varepsilon} \frac{d\mu(w)}{(w-z)^2} \quad (z \in \mathbf{C}).$$

We set

$$(B_*\mu)(z) = \sup_{\varepsilon>0} |(B_\varepsilon \mu)(z)| \quad (z \in \mathbb{C}).$$

We also set

$$(B\mu)(z) = \lim_{\varepsilon \to 0^+} (B_\varepsilon \mu)(z), \tag{3.1}$$

for those $z \in \mathbb{C}$ for which this limit exists. The function $B\mu$ is called the Beurling transform of μ. It is clear from the above definitions that

$$|(B\mu)(z)| \leq (B_*\mu)(z),$$

for all z for which $(B\mu)(z)$ is defined.

The following result is a special case of an important result of Calderon-Zygmund theory (Stein 1970).

PROPOSITION 3.1 (i) *Let $f \in L^1(\mathbb{C}, dm)$ and set $\mu = f\,dm$. Then the limit in (3.1) exists a.e. with respect to m.*

(ii) *B_* maps $L^1(\mathbb{C}, dm)$ into $L^1_*(\mathbb{C}, dm)$, and there exists a constant C_1 with $\|B_* f\| \leq C_1 \|f\|_1$, $(f \in L^1(\mathbb{C}, dm))$.*

NOTATION. In this section, we use the notation $\mu \perp m$ to mean that the measures μ, m are mutually singular.

LEMMA 3.2 (Feinstein 1989) *Let μ be a finite, positive Borel measure on $\mathbb{C}$. Suppose that $\mu \perp m$. Then $B_*\mu \in L^1_*(\mathbb{C}, dm)$. Furthermore, there is a constant $C_2 > 0$ which does not depend on μ with $\|B_*\mu\|_* \leq C_2 \|\mu\|$.*

We shall need the following theorems later.

THEOREM 3.3 (Feinstein 1989) *There is a constant $C_3 > 0$ such that*

$$B_*\mu \in L^1_*(\mathbb{C}, dm) \text{ and } \|BH_*\mu\| \leq C_3 \|\mu\| \quad (\mu \in \mathcal{M}(\mathbb{C})).$$

THEOREM 3.4 (Feinstein 1989) *Let $\mu \in \mathcal{M}(\mathbb{C})$. Then the limit in (3.1) exists a.e. with respect to m, and the Beurling transform maps $\mathcal{M}(\mathbb{C})$ continuously into $L^1_*(\mathbb{C}, dm)$.*

REMARK. Let X be a compact plane set of which $R(X) \neq C(X)$, and let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X, and annihilating $R(X)$. It is standard that $\hat{\mu} = 0$ iff X, and that is not true that $\hat{\mu} = 0$ a.e. with respect to m on $\mathbb{C}$ (see (Browder 1969). Thus $\hat{\mu}|x$ is a non-zero element of $L^1_*(X, dm)$.

Note that this remark is also true for the uniform algebras $P(X)$ and $A(X)$ instead of $R(X)$.

The statement of equality in the next lemma is due to O'Farrell. It can be found on page 379 of (Farrell 1986).

LEMMA 3.5 *Let X be a compact plane set of which $R(X) \neq C(X)$, and let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X, and annihilating $R(X)$. Let $f \in R_0(X)$, Then*

$$f'(z)\hat{\mu}(z) = B(f\mu)(z) - f(z)(B\mu)(z),$$

and

$$|f'(z)\hat{\mu}(z)| \leq B_*(f\mu)(z) + |f(z)|B_*\mu)(z)$$

a.e. with respect to m on X.

REMARK. Lemma 3.5 is true if we use the uniform algebra $P(X)$ instead of $R(X)$ since $P(X) \cong R(\hat{X})$, where $\hat{X}$ is the polynomially convex hull of X.

Similarly, the next theorems are still true for $P(X)$.

Now, we shall use $A(X)$ instead of $R(X)$ in the next there theorems. We have proved these theorems are still true.

THEOREM 3.6 *Let X be a compact plane set for which $A(X) \neq C(X)$, and suppose that $m(\partial X) = 0$ (∂X is the boundary of X). Let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X, and annihilating $A(X)$. Let $f \in A(X)$. Then*

$$f'(z)\hat{\mu}(z) = B(f\mu)(z) - f(z)(B\mu)(z),$$

and

$$|f'(z)\hat{\mu}(z)| \leq B_*(f\mu)(z) + |f(z)|(B_*\mu)(z)$$

a.e. with respect to m on X.

Proof: We set

$$E = \{z \in \text{int}(X) : \tilde{\mu}(z) < \infty\}.$$

It is standard that $\tilde{\mu}(z) < \infty$ a.e. with respect to m. (This follows from the fact that $\tilde{\mu} \in L^1_{log}(\mathbb{C}, dm)$). By the comment immediately following the definition of $\tilde{\mu}$, $|\mu|(\{z\}) = 0$ $(z \in E)$.

Let $z_0 \in E$. Then there exists $g \in A(X)$ with

$$g(w) = \frac{f(w) - f(z_0) - (w - z_0)f'(z_0)}{(w - z_0)^2} \quad (w \in X\backslash\{z_0\}).$$

Clearly $g \in L^1(X, d|\mu|)$. Now, we obtain

$$\begin{aligned} \frac{1}{\pi}\int_X \left|\frac{f'(z_0)}{w - z_0}\right| d|\mu|(w) &= \frac{1}{\pi}|f'(z_0)| \int_X \frac{1}{|w - z_0|} d|\mu|(w) \\ &= \frac{1}{\pi}|f'(z_0)| \int_{\mathbb{C}} \frac{1}{|w - z_0|} d|\mu|(w). \end{aligned}$$

Since $z_0 \in E$, it follows that

$$\frac{1}{\pi}\int_X \left|\frac{f'(z_0)}{w-z_0}\right| d|\mu|(w) = \frac{1}{\pi}|f'(z_0)|\tilde{\mu}(z_0) < \infty.$$

Hence $f'(z_0)/(w-z_0) \in L^1(X, d|\mu|)$.

Then $\frac{1}{\pi}\int_X g(w)d\mu(w) = 0$ because $\mu \perp A(X)$. We obtain

$$\frac{1}{\pi}\int_X \frac{f'(z_0)}{w-z_0} d\mu(w) = f'(z_0)\hat{\mu}_0(z_0).$$

Since $|\mu|(\{z_0\}) = 0$, μ is regular, and μ is supported on X, it follows that

$$(f(w)-f(z_0))/(w-z_0)^2 \in L^1(X, d|\mu|),$$

and that

$$\begin{aligned} f'(z_0)\hat{\mu}(z_0) &= \frac{1}{\pi}\int_X \frac{f(w)-f(z_0)}{(w-z_0)^2} d\mu(w) \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\pi}\int_{X\setminus\Delta(z_0,\varepsilon)} \frac{f(w)-f(z_0)}{(w-z_0)^2} d\mu(w) \\ &= \lim_{\varepsilon\to 0+} (B_\varepsilon(f\mu)(z_0) - f(z_0)(B_\varepsilon\mu)(z_0)). \end{aligned}$$

It is now clear that

$$f'(z)\hat{\mu}(z) = B(f\mu)(z) - f(z)(B\mu)(z)$$

a.e. with respect to m on E, and that

$$|f'(z)\hat{\mu}(z)| \le B_*(f\mu)(z) + |f(z)|(B_*\mu)(z) \quad (z \in E).$$

Since $m(X\setminus E) = 0$, the result now follows.

THEOREM 3.7 *Let X be a compact plane set such that $A(X) \neq C(X)$, and suppose that $m(\partial X) = 0$. Let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X such that $\mu \perp A(X)$, and set $h = \hat{\mu}|_X$. Then the map*

$$D : f \mapsto f' \cdot h, \quad A(X) \to L^1_*(X, dm)$$

is a non-zero, continuous linear operator, and the extension $\tilde{D}$ of D to $A(X)$ is a non-zero, continuous derivation, given by

$$\tilde{D}(f) = B(f\mu)|_X - f\cdot(B\mu)|_X \quad (f \in A(X)).$$

Proof: Let $f \in A(X)$. Then

$$\begin{aligned}\|Df\| &\leq \|B_*(f\mu) + |f|(B_*\mu)\| \\ &\leq 2(\|B_*(f\mu)\| + \||f|(B_*\mu)\|) \\ &\leq 2(C_3\|F\mu\| + C_3\|F\|\,\|\mu\|) \\ &\leq 2(C_3\|f\|\,\|\mu\| + C_3\|f\|\,\|\mu\|) \\ &= 4C_3\|\mu\|\,\|f\|.\end{aligned}$$

So $\|Df\| \leq 4C_3\|\mu\|\,\|f\|$ and hence D is continuous. The continuity of the linear operator $\bar{D}$, and the fact that $\bar{D}$ extends D are immediate consequences of Theorem 3.6, Theorem 3.3 and Theorem 3.4.

The derivation identity is obvious. Since $D(z) = h$, a non-zero element of $L^1_*(X, dm)$, the result is proved.

NOTATION. Let X be a compact subset of $\mathbb{C}$. Then Z denotes the coordinates functional on X which is given by

$$Z(\lambda) = \lambda \quad (\lambda \in \mathbb{C}).$$

In the next theorem, we must distinguish between z which is an element in X and Z for the coordinate functional.

THEOREM 3.8 *Let X be a compact plane set such that $A(X) \neq C(X)$, and suppose that $m(\partial X) = 0$. Let μ be a non-zero element of $\mathcal{M}(\mathbb{C})$ supported on X such that $\mu \perp A(X)$. Then there exists $C > 0$ with*

$$\inf\{|f'(z)| : z \in X\} \leq C|f|_X \quad (f \in A(X)). \tag{3.2}$$

Proof: Let μ and D be as in Theorem 3.7. Then

$$D(f) = f' \cdot D(Z) \quad (f \in A(X)).$$

Set

$$Q = \{D(f) : f \in A(X),\ |f|_X \leq 1\}.$$

Clearly Q is a bounded subset of $L^1_*(X, dm)$. Suppose, for a contradiction, that (3.2) is not satisfied for any $C > 0$. Let (f_n) be a sequence in $A(X)$ satisfying

$$f'_n|_X \leq 1, \quad \inf\{|f'_n(z)| : z \in X\} \geq n \quad (n \in \mathbb{N}).$$

and set

$$g_n = 1/f'_n \quad (n \in \mathbb{N}).$$

Then $g_n \to 0$ uniformly on X, and so

$$D(Z) = g_n \cdot f_n' \cdot D(Z) = g_n \cdot D(f_n) \to 0 \text{ as } n \to \infty.$$

Thus $D(Z) = 0$, a contradiction of Theorem 3.7.

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