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FIXED POINTS ON 2-METRIC SPACES

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In this paper, we give some result in the area of uniqueness of fixed points in the case of complete 2-metric space.

INTRODUCTION

The concept of a 2-metric space was introduced by S. Gabler [3], Recently the 2-metric spaces has been developed extensively in different subjects by others, for example [5], [6] and [7].

Definition 1.1. Let F be a function from a non-empty set X into itself such that $Fx = x$. Then x in X is called a fixed point of F.

Definition 1.2. A 2-metric space is a space X in which for each triple points x, y, z there exists with a real function d defined on $X \times X \times X$ such that

(i) to each pair of points x, y with $x \neq y$ from X, there is $z \in y$ such that $d(x, y, z) \neq 0$.

(ii) $d(x, y, z) = 0$ only when at least two of three points are equal.

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$.

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.

Remark. Let $y = z$ in (iv). Then we have

$$0 \leq d(x, y, u) + d(x, u, y) = 2d(x, y, u).$$

Hence d is non-negative.

Definition 1.3. Let X be a 2-metric space. If $d(x_n, x, a) \to 0$ for all $a \in X$, we say that the sequence $\{x_n\}$ converges to x, and x is a limit of $\{x_n\}$.

Definition 1.4. Let X be a 2-metric space. If $d(x_n, x_m, a) \to 0$ $(m, n \to \infty)$ for all $a \in X$, we say that the sequence $\{x_n\}$ is called a Cauchy sequence.

Definition 1.5. Let X be a 2-metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete 2-metric space.

SOME RESULTS ON FIXED POINT OF 2-METRIC SPACE

We give some result in the area of uniqueness of fixed points of complete 2-metric spaces.

Let us start with the following definition.

Definition 2.1. Let F be mapping of 2-metric space X into itself. If for all $a \in X$

$$d(F^{n_i}x,\ u,\ a) \to 0\ (i \to \infty)$$

implies $$d(FF^{n_i}x,\ fu,\ a) \to 0\ \ (i \to \infty),$$

then F is called orbitally continuous.

The following theorem will be use in the next our theorem.

Theorem 2.1. [3]. Let F be a orbitally continuous mapping of a 2-metric space X into itself. If x_0 is the limit of $F^n x$ for some $x \in X$, then x_0 is a fixed point of F.

Theorem 2.2. Let F be a orbitally continuous mapping from a complete 2-metric space X into itself satisfies for all $a, x, y \in X$

$$d(Fx, Fy,\ a) \le a_1 d(x, Fx,\ a) + a_2 d(y,\ Fy\ a) + a_3 d(x,\ y,\ a),$$

where a_1, a_2 and a_3 are non-negative numbers such that

$$a_1 + a_2 + a_3 < 1.$$

Then F has the unique fixed point u in X.

Proof. Let $x_0 \in X$. We define the sequence $\{x_n\}$ by

$$x_{n+1} = Fx_n \qquad (n = 0,\ 1,\ 2,\ \ldots)$$

Then

$$\begin{aligned} d(x_n,\ d_{n+1},\ a) &\le d(Fx_{n-1},\ Fx_n, a) \\ &\le a_1 d(x_{n-1},\ Fx_{n-1},\ a) + a_2 d(x_n,\ Fx_n,\ a) + a_3 d(x_{n-1},\ x_n,\ a) \\ &= a_1 d(x_{n-1},\ x_n, a) + a_2 d(x_n,\ x_{n+1},\ a) + a_3 d(x_{n-1},\ x_n,\ a). \end{aligned}$$

Therefore

$$(1 - a_2)\, d(x_n,\ x_{n+1}, a) \le (a_1 + a_3)\, d(x_{n-1}, x_n,\ a).$$

So

$$d(x_n,\ x_{n+1}, a) \le \frac{a_1 + a_3}{1 - a_2} d(x_{n-1},\ x_n,\ a).$$

In the same way, we can obtain

$$d(x_n, x_{n+1}, a) \le \left(\frac{a_1 + a_3}{1 - a_2}\right)^n d(x_0,\ x_1).$$

It follows that $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) \le \sum_{n=0}^{\infty} \left(\frac{a_1 + a_3}{1 - a_2}\right)^n d(x_0,\ x_1)$. Since $\frac{a_1 + a_3}{1 - a_2} < 1$, so the sequence $\{x_n\}$ is Cauchy. By completeness of X the sequence $\{x_n\}$ converges to the element u in X. That is $\lim_{n \to \infty} x_n = u$.

We have for all $a \in X$

$$\lim_{n \to \infty} d(F^n x_0,\ u,\ a) = 0.$$

Since F is orbitally continuous, so

$$\lim_{n\to\infty} d(F^{n+1}x_0, Fu, a) = 0$$

Therefore

$$d(u, Fu, a) \le d(u, Fu, F^{n+1}x_0) + d(u, F^{n+1}x_0, a) + d(F^{n+1}x_0, Fu, a) \to 0 (n \to \infty).$$

Hence $d(u, Fu, a) = 0$. If $u \ne Fu$, then $d(u, Fu, a) \ne 0$ for some a. This implies $u = Fu$. So u is a fixed point of F.

For uniqueness, let w be another fixed point of F. Then

$$d(u, w, a) = d(Fu, Fw, a)$$
$$\le a_1 d(u, Fu, a) + a_2 d(w, Fw, a) + a_3 (u, w, a).$$

Therefore

$$d(u, w) \le a_3 d(u, w).$$

Thus $u = w$.

This completes the proof.

We state and prove some consequences of Theorem 2.2.

Corollary 2.1. Let F be a orbitally continuous mapping from a complete 2-metric space X into itself and $a, x, y \in X$. Let F_k $(k = 1, , ..., n)$ be a family of mappings from X into itself. Let

$$F_k F_j = F_j F_k \ (k, j = 1, 2, ..., n)$$

and $d(F_1 F_2 ... F_n x, F_1 F_2 ... F_n y, a) \le a_1 d(x, F_1 F_2 ... F_n x, a) + a_2 d(y, F_1 F_2 ... F_n y, a) + a_3 d(x, y, a)$.

where a_1, a_2 and a_3 are non-negative numbers such that $a_1 + a_2 + a_3 < 1$.

Then F_k have the unique common fixed point in X.

Proof. Set $V = F_1 F_2 ... F_n$. By Theorem 2.2, we have

$$d(Vx, Vy, a) \le a_1 d(x, Vx, a) + a_2 d(y, Vy, a) + a_3 d(x, y, a)$$

Then V has the unique common fixed point in X, say x'. So $Vx' = x'$. Thus

$$V(F_1 x') = F_i (Vx') (i = 1, 2, ..., n)$$
$$= F_i(x').$$

Hence $F_i(x')$ is a fixed point of V. Since V has the unique fixed point x', it follows that $F_i(x') = x'$. This x' is common fixed point of F_i. For uniqueness, let x', x'' be fixed points of V Then

$$d(x', x'', a) = d(Vx', Vx'', a)$$
$$\le a_1 d(x', Vx', a) + a_2 d(x'', Vx'', a) + a_3 d(x', x'', a)$$

It follows that

$$d(x', x'', a) \le a_3 d(x', x'', a).$$

So $\qquad x' = x''.$

This completes the proof.

Corollary 2.2. Let $F_k (k = 1, 2, ..., n)$ be orbitally continuous mapping from a complete 2-metric space X into itself and $a, x, y \in X$. Let $F_k F_j = F_j F_k$ $(k, j = 1, 2, ..., n)$. Suppose there a system of positive integers $m_1, m_2, ..., m_n$ such that

$$d(F_1^{m_1}, ..., F_1^{m_1} F_2^{m_2}, ..., F_n^{m_n} y, a) \leq a_1 d(x, F_1^{m_1} F_2^{m_2}, ..., F_n^{m_n} x, a)$$

$$+ a_2 d(y, F_1^{m_1} F_2^{m_2}, ..., F_n^{m_n}, ..., F_n^{m_n} y, a) + a_3 d(x, y, a).$$

where a_1, a_2 and a_3 are non-negative numbers such that $a_1 + a_2 + a_3 < 1$.

Then F_k have the unique common fixed point in X.

Proof. Let $U = F_1^{m_1} F_2^{m_2} F_n^{m_n}$. Then the proof follows by Theorem 2.2.

Corollary 2.3. Let $F_k (k = 1, 2, ..., n)$ be a family orbitally continuous mapping from a complete 2-metrick space X into itself and $a, x, y \in X$. Let $F_1 F_2 ... F_n$ be commutes with every $F_i (i = 1, 2, , ..., n)$. Suppose that

$$d(F_1 F_2 ... F_n x, F_n F_{n-1} ... F_1 y, a) \leq a_1 d(x, F_1 F_2 ... F_n x, a)$$

$$+ a_2 d(y, F_n F_{n-1} ... F_1 y, a) + a_3 d(x, y, a),$$

where a_1, a_2 and a_3 are non-negative numbers such that $a_1 + a_2 + a_3 < 1$.

Then F_k have the unique common fixed point in X.

Proof. Let $U = F_1 F_2, ..., F_n, V = F_n F_{n-1}, ..., F_1$. Then

$$d(Ux, Vy, a) \leq a_1 d(x, Ux, a) + a_2 d(y, Vy, a) + a_3 d(x, y, a).$$

By theorem 2.2, U and V have the unique common fixed point x in X. Then

$$U * x = V_X = x.$$

For any F_i we have $F_i(Ux) = F_i(x)$. By the assumption,

$$U(F_i(x)) = F_i(x) \text{ and } V(F_i(x)) = F_i(x).$$

Hence $F_i(x) = x (i = 1, 2, ..., n)$ which means that x is a common fixed point of $\{F_i\}$. Clearly x is the unique common fixed point of $\{F_i\}$.

This completes the proof.

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