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Three kinds of Uniform Algebras

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Abstract

Assume X is a compact space. It is possible to construct some different uniform algebras on X. In this paper, we investigate three kinds of related uniform algebras on compact space X. They are normal, have bounded relative units and strongly regular uniform algebras. We study some results and some connections between the previous by mentioned uniform algebras.

الملخص

نفرض ان X فراغ مضغوط . يمكننا انشاء او تكوين بعض المجموعات الجبرية المتنظمة المختلفة على X . فى هذا البحث سوف ندرس ثلاثة انواع من المجموعات الجبرية المتنظمة المرتبطة على X و هى : الجبر الطبيعى , جبر الوحدات النسبية المحدودة و الجبرالمنتظم بقوة. سوف نعطى بعض النتائج و العلاقات التى تخص هذه المجموعات الجبرية المتنظمة السابقة الذكر.

1. Introduction

In this section, we shall give some standard definitions and results which we shall need later in this paper.

Definition 1.1. Let X be a non-empty set, and let f be a bounded complex-valued function on X .The *uniform norm* of f on X, denoted by $|f|_X$, is defined by

$$|f|_X = \sup\{|f(x)| : x \in X\}.$$

Definition 1.2. An *algebra* A over the set of all complex numbers $\mathbb{C}$ is a vector space A over $\mathbb{C}$ such that for all x, y and $z \in A$, $\alpha \in \mathbb{C}$ the following conditions are satisfied :

$$x(yz) = (xy)z$$

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$$(x+y)z = xz + yz$$
$$x(y+z) = xy + xz$$
$$\alpha(xy) = x(\alpha y) = (\alpha x)z.$$

If an algebra A has an identity, then we shall denote this identity by e.

Definition 1.3. A *normed algebra* is an algebra A with unit e which is normed as a linear space and in which

$$||ab|| \le ||a||\,||b|| \quad (a, b \in A).$$

We assume $\|e\| = 1$.

Definition 1.4. A *Banach algebra* A is a complete normed algebra A.

Definition 1.5. [4] Let A be a Banach algebra. A linear map $\phi : A \to \mathbb{C}$ is called a *character* if $\phi \neq 0$ and

$$\phi(ab) = \phi(a)\,\phi(b) \quad (a, b \in A),$$

that is ϕ is a non-zero multiplicative linear functional on A.

Theorem 1.1 [4]. *Let ϕ be a character mapping on a Banach algebra A. Then ϕ is continuous and $||\phi|| = 1$.*

Notation. Let A be a commutative Banach algebra. Then Φ_A denotes the set of all character mappings on A.

Definition 1.6. Let A be a commutative Banach algebra. Let $\phi \in \Phi_A$. The *kernel* of ϕ in A, is denoted by $\ker(\phi)$, defined by

$$\ker(\phi) = \{ f \in A : \phi(f) = 0 \}.$$

Theorem 1.2 [1]. *Let A be a commutative Banach algebra. Let ϕ_1, ϕ_2 be two character mappings on A. If $\ker(\phi_1) \subseteq \ker(\phi_2)$, then $\phi_1 = \phi_2$.*

Throughout this paper, a compact space X is a compact Hausdorff topological space.

Notation. Let X be a compact space. We shall denote by $C(X)$ the algebra of all continuous functions from X into $\mathbb{C}$.

Then $C(X)$ is a commutative Banach algebra with the uniform norm.

Definition 1.7. Let A be a subset of $C(X)$. Then A is said be *separates the points* of X if for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ with $f(x) \neq f(y)$.

Definitions 1.8. Let X be a compact space. A *uniform algebra* on X is a closed subalgebra of $C(X)$ which contains the constant functions, and separates the points of X.

Let us note that every uniform algebra is a commutative Banach algebra with unit.

Definition 1.9.. Let A be a uniform algebra on a compact space X. For each $x \in X$ the mapping ε_x, defined by

$$\varepsilon_x(f) = f(x) \quad (f \in A),$$

is called the *evaluation mapping* at x.

Cleary $\varepsilon_x \in \Phi_A$ (that is, every evaluation mapping is a character).

Definitions 1.10.[5] Let A be a uniform algebra on a compact space X. Let $x \in X$. We define the following ideals in A by setting

$$J_x = \{ f \in A : f^{-1}(0) \text{ is a neighbourhood of } x \}.$$

$$M_x = \{ f \in A : f(x) = 0 \},$$

and

$$M_x^2 = \left\{ \sum_{i=1}^{n} f_i g_i \ : \ f_i, g_i \in M_x \right\}.$$

We notice that $M_x = \ker(\varepsilon_x)$ and M_x is a maximal ideal.

Lemma 1.3 [5]. *Let A be a uniform algebra on a compact space X. Let $x \in X$. Then M_x is closed in A.*

Remarks. We shall record the following observations

(i) $J_x \subseteq M_x$ and $M_x^2 \subseteq M_x$.

(ii) M_x^2 is not necessarily closed in A.

(iii) Since M_x is closed in A and since $\overline{J_x}$ (the closure of J_x) is the smallest closed subset containing J_x, so $J_x \subseteq \overline{J_x} \subseteq M_x$.

Theorem 1.4 (Urysohn's Lemma) [5]. *Let E and F be the disjoint non-empty closed subsets of a compact space X. Then there exists a continuous function $f : X \to [0,1]$ such that*

$$f(x) = 0 \quad \text{for all } x \in F,$$

and

$$f(x) = 1 \quad \text{for all } x \in E.$$

Definition 1.11. Let A be a uniform algebra on a compact space X. Then A is called *trivial* if $A = C(X)$. Otherwise, it is *non-trivial.*

Definition 1.12. Let A be a uniform algebra on a compact space X and $x \in X$. Let $\phi \in \Phi_A$. Then a *Jensen measure* for ϕ is a regular measure μ on X satisfying

$$\log|\phi(f)| \leq \int_X \log|f(x)| \, d\mu(x) \quad (f \in A).$$

Theorem 1.5.[2] *Let A be a uniform algebra on a compact space X. Let $\phi \in \Phi_A$. Then*

(i) *There exists a Jensen measure for ϕ.*

(ii) *Every Jensen measure for ϕ is a representing measure for ϕ.*

Notation. Let (X, d) be a metric space. The open disc $\Delta(a,r)$ with centre a and radius r, is defined by

$$\Delta(a,r) = \{x \in X : d(x,a) < r\}.$$

The closed disc $\overline{\Delta}(a,r)$ is given by

$$\overline{\Delta}(a,r) = \{x \in X : d(x,a) \leq r\}.$$

2. Normal, having bounded relative units and strongly regular uniform algebras

In this section, we shall give three kinds of related uniform algebras. They are the normal, having bounded relative units and the strongly regular algebras. We have induced some results concerning these uniform algebras.

Now, we state and prove the following lemma.

Lemma 2.1. *Let A be a uniform algebra on a compact space X. Let $x \in X$. Then M_x separates the points of X.*

Proof. Let $x, y \in X$ and $x \neq y$. Let A be a uniform algebra on a compact space X. Then there exists $f \in A$ with $f(x) \neq f(y)$.

Let x_0 be a fixed in X. Define a function g by

$$g(w) = f(w) - f(x_0) \quad (w \in X).$$

Let $w_1, w_2 \in X$ such that $w_1 = w_2$. Then $f(w_1) = f(w_2)$. So

$$f(w_1) - f(x_0) = f(w_2) - f(x_0).$$

Hence $g(w_1) = g(w_2)$. Thus g is a well-defined function.

Clearly $g \in M_X$. Then

$$g(x) - g(y) = f(x) - f(y) \neq 0.$$

Hence $g(x) \neq g(y)$. Thus M_x separates the points of X.

Definition 2.1 [6]. Let X be a compact subset of $\mathbb{C}$. Then $R_0(X)$ is the set of restricted functions on X of rational functions with poles off X.

We define $R(X)$ is the closure of $R_0(X)$ in $C(X)$.

The following lemma is standard.

Lemma 2.2. Let X be a compact subset of $\mathbb{C}$. *Then $R(X)$ is a uniform algebra on X.*

Definition 2.2 [6]. Let A be a uniform algebra on a compact space X. Then A is called *normal on* X if for each pair of non-empty closed sets E,F contained in X with $F \cap E = \varnothing$, there exists $f \in A$ with $f(E) \subseteq \{1\}$ and $f(F) \subseteq \{0\}$.

We state some results concerning normal uniform algebras.

Theorem 2.3 [4]. Let X be a compact subset of $\mathbb{C}$. *If* $R(X)$ *is a normal uniform algebra, then the interior of* X *is an empty set.*

Theorem 2.4 [3]. Let X be a compact subset of $\mathbb{C}$. *Let* X_i $(i = 1,2,\ldots,n)$ *be compact spaces and* $X = \bigcup_{i=1}^{n} X_i$. *If* $R(X_i)$ *is normal on* X_i, *then* $R(X)$ *is normal on* X.

Theorem 2.5 [6]. *Let* A *be a normal uniform algebra on a compact space* X. *Let* $x \in X$. *Then* J_x *separates the points of* X.

Remark. Let A be a normal uniform algebra on a compact space X. Let $x \in X$. If we drop the normality condition of A, then J_x may not be separates the points of X. For example, let $X = \overline{\Delta}(0,1)$ be the closed unit disc. Then $R(\overline{\Delta}(0,1))$ is a uniform algebra on the compact space $\overline{\Delta}(0,1)$. Let $f \in J_x$. Then we have the following related steps

f is a holomorphic function on $\Delta(0,1)$

$f = 0$ on a neighbourhood of $x \in X$

$f = 0$ on $\Delta(0,1)$ (by the identity principle).

$f = 0$ on $\overline{\Delta}(0,1)$ (by continuity of f).

Thus $$J_x = \left\{ f \in R(\overline{\Delta}(0,1)) : f = 0 \text{ on } \overline{\Delta}(0,1) \right\}$$

$$= \{0\}.$$

Hence J_x does not separate the points of $R(\overline{\Delta}(0,1))$.

Lemma 2.6 [6]. *Let* A *be a normal uniform algebra on a compact space* X, *and let* $x \in X$. *Then* $J_x \subseteq M_x^2$.

Definition 2.3. Let A be a uniform algebra on a compact space X, and let $x \in X$. Then A has *bounded relative units at* x with bound C if for each compact subset E of $X \setminus \{x\}$, there exists $f \in J_x$ with $f(E) \subseteq \{1\}$ and with $|f|_X \leq C$.

The uniform algebra A on a compact space X has *bounded relative units* if A has bounded relative units at every point x of X.

We give the following example.

Example 2.1. Let X be a compact space. Let E be a closed subset of X, and let $x \in X \setminus E$. Let F be a closed neighbourhood of x in X such that $E \cap F = \varnothing$.. By Urysohn's lemma, there a function $f \in C(X)$ such that $f : X \to [0,1]$,

$$f = 0 \quad \text{on} \quad F,$$

and

$$f = 1 \quad \text{on} \quad E.$$

Then $|f|_X \leq 1$ and $f \in J_x$.

Thus $C(X)$ has bounded relative units.

Theorem 2.7 [3]. Let X be a compact subset of $\mathbb{C}$. *Let A be a uniform algebra on X. If $R(X)$ has bounded relative units, then $C(X) = R(X)$.*

Theorem 2.8 [3]. *There exists a non-trivial uniform algebra A on a compact space X such that A has bounded relative units.*

Theorem 2.9 [3]. *Let A be a normal uniform algebra on a compact space X. Suppose that $A = \{ f^2 : f \in A \}$. Then A has bounded relative units.*

There are two different methods for the proof of the next theorem. We shall give the technique of the first method of this proof. The second method of the proof appears later in this paper.

Theorem 2.10. *Let A be a uniform algebra on a compact space X. If A has bounded relative units, then A is normal.*

Proof. Suppose A has bounded relative units. Let E, F be two closed disjoint subsets of X. Then E, F are compact of X. Let $x \in F$ and let u_x be a neighbourhood of x. Since $x \notin E$, so there a function $f_x \in J_x$ such that $f_x(u_x) = 0$, $f_x(E) = 1$

and $|f|_X \leq C$ Since $\{u_x : x \in F\}$ cover F, so there exist finite open subcover

$\{u_{x_1}, \ldots, u_{x_n} : x \in F\}$ of F. Let $f = \prod_{i=1}^{n} f_{x_i}$. Then

$$f_x(u_{x_i}) = 0 \quad \text{for all} \quad i = 1, \ldots, n.$$

Hence $f(F) = 0$. Also $f(E) = 1$.

This completes the proof.

Definition 2.4 [3]. Let A be a uniform algebra on a compact space X, and let $x \in X$. Then A is called *strongly regular at* x if $\overline{J_x} = M_x$.

The uniform algebra A on a compact space X is *strongly regular* if it is strongly regular at every point $x \in X$.

Remark. There are some uniform algebras on compact spaces which are not strongly regular. For example, let $n \in \mathbb{N}$, and let $I = [a,b]$ be a compact interval of $\mathbb{R}$.

Let $C^{(n)}(I)$ be the algebra of n-times continuously differentiable functions on I with the norm

$$\| f \|_n = \sum_{k=0}^{n} \frac{1}{k!} |f^{(k)}|_I \quad (f \in C^{(n)}(I))$$

Let $n \geq 1$. Then for $k = 0, 1, \ldots, n$, we can define

$$M_{n,k}(x_0) = \{ f \in C^{(n)}(I) : f^{(j)}(x_0) = 0 \ (j = 0, 1, 2, \ldots, k) \}.$$

Each $M_{n,k}(x_0)$ is a closed ideal in $C^{(n)}(I)$ and

$$M_{n,n}(x_0) \subseteq M_{n,n-1}(x_0) \subseteq \ldots \subseteq M_{n,1}(x_0) \subseteq M_{n,0}(x_0).$$

The ideal $M_{n,0}(x_0)$ is the maximal ideal at x_0. Then

$$J_{x_0} \subseteq M_{n,n}(x_0) \underset{\neq}{\subset} M_{n,0}(x_0) = M_{x_0}$$

Thus $\overline{J_{x_0}} \subseteq M_{n,n}(x_0) \underset{\neq}{\subset} M_{x_0}$.

It follows that $\overline{J_{x_0}} \neq M_{x_0}$. Hence $C^{(n)}(I)$ is not a strongly regular uniform algebra.

The following results concerning strongly regular uniform algebras.

Theorem 2.11 [3] . *Let A be a uniform algebra on a compact space X. If A has bounded relative units, then A is strongly regular.*

Theorem 2.12 [7] . *If A is strongly regular uniform algebra on a compact space X, then A is normal.*

Corollary 2.13 *Let A be a uniform algebra on a compact space X, If A has bounded relative units, then A is normal.*

Proof. The proof follows by Theorem 2.11 and Theorem 2.12 .

Theorem 2.14 . Let X be a compact subset of $\mathbb{C}$. *Let $R(X)$ be a strongly regular uniform algebra on a compact space X. Let $x \in X$. Then J_x separates the points of X.*

Proof. The proof follows by Theorem 2.12. and Theorem 2.5.

Theorem 2.15 . Let X be a compact subset of $\mathbb{C}$. *If $R(X)$ is a strongly regular uniform algebra, then the interior of X is an empty set*

Proof. The proof follows by Theorem 2. 12 and Theorem 2.3.

We state and prove the following two theorems.

Theorem 2.16. . *Let A be strongly regular uniform algebra on a compact space X. Then $M_x = \overline{M_x^2}$.*

Proof. . Let A be strongly regular uniform algebra on a compact space X. Then A is normal (by Theorem 2. 13). So by Lemma 2. 6 , $J_x \subseteq M_x^2$.

Hence $\overline{J_x} \subseteq \overline{M_x^2}$. Since A is strongly regular, it follows that $M_x \subseteq \overline{M_x^2}$

Since $M_x^2 \subseteq M_x$ and M_x is closed , so $\overline{M_x^2} \subseteq M_x$. Thus $M_x = \overline{M_x^2}$.

Definition 2.5. Let X be a compact space and let $H \subseteq X$. Then H is called the *closed support* of a measure μ on X if H is the smallest closed set such that $\mu(X \setminus H) = 0$.

Theorem 2.17. *Let A be a uniform algebra on a compact infinite space X. Suppose that A is strongly regular at all but finitely many points of X. Let $\phi \in \Phi_A$. Then $\phi = \varepsilon_x$.*

Proof. Let $\phi \in \Phi_A$, and let μ Jensen measure for ϕ on X. Let H be the closed support of μ. Suppose that A is strongly regular at some point $x \in H$.

Let $f \in J_x$. Since f is identically zero in a neighbourhood of $x \in H$., so we obtain

$$\int_H \log|f(x)|\, d\mu(x) = -\infty .$$

Since

$$\int_X \log|f(x)|\, d\mu(x) = \int_H \log|f(x)|\, d\mu(x) + \int_{X\setminus H} \log|f(x)|\, d\mu(x),$$

and $\mu(X \setminus H) = 0$, it follows that

$$\int_{X\setminus H} \log|f(x)|\, d\mu(x) = 0.$$

Hence $\log|\phi(f)| \leq \int_X \log|f(x)|\, d\mu(x) = -\infty.$

Thus $\phi(f) = 0$. Hence $J_x \subseteq M_\phi$. We have $\overline{J_x} = M_x$. It follows that $J_x \subseteq \overline{J_x} \subseteq M_\phi$. Thus $J_x \subseteq M_x \subseteq M_\phi$. Hence $M_x \subseteq M_\phi$. So $\ker(\varepsilon_x) \subseteq \ker(\phi)$.

Hence By Theorem 1.2, $\phi = \varepsilon_x$.

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