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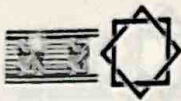
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***A study of properties  
of Root extensions***

**By  
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## A Study of properties of Root Extensions

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### دراسة لخواص الامتدادات الجذرية

خلاصة

نفرض ان  $A$  هو الجبر المنتظم على الفراغ المتراص  $X$  ونفرض ان  $F$  فئة جزئية من  $A$ . نفرض ان  $\tilde{A}$  عبارة عن  $F$ -امتداد من  $A$ . فان  $\tilde{A}$  هو الامتداد الجذري للجبر المنتظم  $A$ . في هذا البحث سوف نقدم دراسة لخواص الامتدادات الجذرية.

### ABSTRACT

Let  $A$  be a uniform algebra on a compact space  $X$  and  $F$  be a subset of  $A$ . Let  $\tilde{A}$  be  $F$ -extension of  $A$ . Then  $\tilde{A}$  is a root extension of  $A$ . In this paper, we give a study of properties of root extensions.

### 1. Introduction

In this section, we shall give some standard definitions and results we shall need later in this paper.

**Definition 1.1 [2].** Let  $S$  be a non-empty set, and let  $f$  be a bounded, complex-valued function on  $S$ . For each non-empty set  $E$  contained in  $S$ , the uniform norm of  $f$  on  $E$ , denoted by  $\|f\|_E$ , is defined by

$$\|f\|_E = \sup\{|f(x)| : x \in E\}.$$



**Definition 1.2.** [1]. The coordinate functional  $Z$  on the complex field  $\mathbb{C}$

defined by

$$Z(\lambda) = \lambda \quad (\lambda \in \mathbb{C}).$$

**Definition 1.3.** [4]. Let  $(X, d)$  be a metric space. Let  $a \in X$ . The closed

disc  $\bar{\Delta}(a, r)$  with centre  $a$  and radius  $r$ , is defined by

$$\bar{\Delta}(a, r) = \{x \in X : d(x, a) \leq r\}.$$

**Lemma 1.1** [4]. A function  $f$  from a topological space  $(X, T)$  into a topological space  $(Y, T')$  is continuous if and only if  $f^{-1}(\{O\})$  is open in  $X$  for every open set  $O$  in  $Y$ .

**Definition 1.4** [4]. A topological space  $(X, T)$  is called a metrizable space if there exists a metric  $d$  on  $X$  which defines the topology  $T$  on  $X$ .

**Lemma 1.2** [4]. A closed subset of a metrizable space is metrizable.

**Lemma 1.3** [4]. A countable product of metrizable spaces is metrizable.

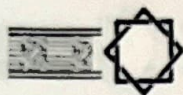
**Lemma 1.4** [4]. A closed subset of a compact space is compact.

**Theorem 1.5** [4]. ( **Tychonoff** ). The product of a collection of compact spaces is compact.

**Definition 1.5** [3]. A normed algebra is an algebra  $A$  which is a normed linear space  $A$  and

$$\|ab\| \leq \|a\| \|b\| \quad (a, b \in A).$$

**Definition 1.6.** [3]. A complete normed algebra  $A$  is called a **Banach algebra**.



**Definition 1.7 [3].** Let  $A$  be a Banach algebra. The non-zero element  $e$  in  $A$  is called an *identity element* (or *unit*) if

$$ea = ae = a \quad (a \in A).$$

If a Banach algebra  $A$  has an identity element  $e$ , then the identity element is unique.

A Banach algebra  $A$  is called *commutative* if

$$ab = ba = e \quad (a, b \in A).$$

In this paper, we assume the identity element  $e = 1$ .

**Definition 1.8. [3].** Let  $A$  be a Banach algebra with unit. The element  $a$  in  $A$  is called an *invertible* if there exists an element  $b$  in  $A$  such that

$$ab = ba = 1.$$

**Lemma 1.6. [3].** Let  $A$  be a Banach algebra with unit. Let  $a$  be an invertible element in  $A$ . Then  $a^2$  is invertible in  $A$ .

Throughout this paper, a compact space  $X$  is a compact Hausdorff topological space.

**Notation.** Let  $X$  be a compact space. We shall denote by  $C(X)$  the Banach algebra of all continuous functions from  $X$  into  $\mathbb{C}$ .

**Definition 1.9. [2].** Let  $F$  be a subset of  $C(X)$ . Then  $F$  said to be *separates the points* of  $X$  if for each  $x, y \in X$  with  $x \neq y$ , there exists  $f \in F$  such that  $f(x) \neq f(y)$ .

**Definition 1.10.[2].** A *uniform algebra*  $A$  on a compact space  $X$  is a closed subalgebra of  $C(X)$  which contains the constant functions, and which separates the points of  $X$ .

**Remark.** Note that every uniform algebra is a commutative Banach algebra with unit.



## 2. Root Extensions

Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $F$  be a subset of  $A$ . We denote by  $\mathbb{C}^F$  the collection of all functions from  $F$  into  $\mathbb{C}$ . We give  $\mathbb{C}^F$  the usual product topology.

**Lemma 2.1.** *Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $F$  be a subset of  $A$ . Then  $\mathbb{C}^F$  is a metrizable space.*

**Proof.** Since  $\mathbb{C}^F = \prod_{f \in F} \mathbb{C}$ , so by Lemma 1.3,  $\mathbb{C}^F$  is metrizable.

Let  $z$  be a coordinate functional. Let  $x \in X$ . We set

$$\tilde{X} = \left\{ (x, z) \in X \times \mathbb{C}^F : z(f)^2 = f(x) \ (f \in F) \right\}.$$

We state and prove the next results concerning the space  $\tilde{X}$ .

**Lemma 2.2.** *Let  $X$  be a compact space. Then*

$$\tilde{X} \subseteq X \times \left\{ z \in \mathbb{C}^F : |z(f)| \leq |f|_X^{\frac{1}{2}} \ (f \in F) \right\}.$$

**Proof.** Let  $(x, z) \in \tilde{X}$ . Then  $z(f)^2 = f(x)$ .

So  $|z(f)|^2 = |f(x)|$ . Thus

$$|z(f)| = \sqrt{|f(x)|}$$

$$\leq \sqrt{|f|_X}$$

This completes the proof.

**Lemma 2.3.** *Let  $X$  be a compact space. Then  $\tilde{X}$  is closed.*



**Proof.** We define a function  $g_f : X \times \mathbb{C}^F \rightarrow \mathbb{C}$  by

$$g_f(x, z) = z(f)^2 - f(x) \quad (f \in F, (x, z) \in X \times \mathbb{C}^F).$$

Then  $g_f$  is a well-defined and continuous function. We have

$$\begin{aligned} \tilde{X} &= \left\{ (x, z) \in X \times \mathbb{C}^F : z(f)^2 = f(x) \quad (f \in F) \right\} \\ &= \bigcap_{f \in F} \left\{ (x, z) \in X \times \mathbb{C}^F : z(f)^2 = f(x) \right\} \\ &= \bigcap_{f \in F} g_f^{-1}(\{0\}). \end{aligned}$$

Since  $g_f$  is continuous, so  $\tilde{X}$  is closed.

**Lemma 2.4.** Let  $X$  be a compact space. Then  $\tilde{X}$  is a compact space.

**Proof.** We shall need to prove that

$$\left\{ z \in \mathbb{C}^F : |z(f)| \leq |f|_X^{\frac{1}{2}} \quad (f \in F) \right\}$$

is compact. We have

$$\begin{aligned} \left\{ z \in \mathbb{C}^F : |z(f)| \leq |f|_X^{\frac{1}{2}} \quad (f \in F) \right\} &= \prod_{f \in F} \left\{ w \in \mathbb{C} : |w| \leq |f|_X^{\frac{1}{2}} \right\} \\ &= \prod_{f \in F} \overline{\Delta} \left( 0, |f|_X^{\frac{1}{2}} \right), \end{aligned}$$

where  $\prod_{f \in F} \overline{\Delta} \left( 0, |f|_X^{\frac{1}{2}} \right)$  is the closed disc with center at 0 and

radius  $|f|_X^{\frac{1}{2}}$ .



Since  $\overline{\Delta}(0, |f|_X^{\frac{1}{2}})$  is compact, so by Tychonoff's Theorem 1.5

$\prod_{f \in F} \overline{\Delta}(0, |f|_X^{\frac{1}{2}})$  is also compact. Again, by Tychonoff's Theorem

$$X \times \left\{ z \in \mathcal{C}^F : |z(f)| \leq |f|_X^{\frac{1}{2}} \quad (f \in F) \right\},$$

is a compact space. Since  $\tilde{X}$  is a closed subset of

$$X \times \left\{ z \in \mathcal{C}^F : |z(f)| \leq |f|_X^{\frac{1}{2}} \quad (f \in F) \right\}$$

so by Lemma 1.4,  $\tilde{X}$  is a compact space.

**Lemma 2.5.** Let  $X$  be a metrizable space. Then  $\tilde{X}$  is a metrizable space.

**Proof.** Let  $X$  be a metrizable space. Then  $X \times \mathcal{C}^F$  is metrizable (Lemma 1.3). Since  $\tilde{X} \subseteq X \times \mathcal{C}^F$  and  $\tilde{X}$  is closed (Lemma 2.3), so  $\tilde{X}$  is metrizable.

Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $F$  be a subset of  $A$ . For each  $f \in F$  we set

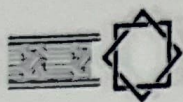
$$p_f(x, z) = z(f) \quad ((x, z) \in \tilde{X})$$

Thus each  $p_f$  is an element of  $C(\tilde{X})$ . Then

$$(p_f(x, z))^2 = f(x) \quad ((x, z) \in \tilde{X}).$$

We define  $\Pi: \tilde{X} \rightarrow X$  by

$$\Pi(x, z) = x \quad ((x, z) \in \tilde{X}).$$



Clearly  $\Pi$  is a continuous mapping. We define  $\Pi^* : C(X) \rightarrow C(\tilde{X})$  by

$$\Pi^*(f) = f \circ \Pi \quad (f \in C(X)).$$

**Lemma 2.6.**  $\Pi^*(1) = 1$ .

**Proof.** There is  $1 \in C(X)$  with  $1(x) = 1$  for all  $x$ . Also, there is  $\tilde{1} \in C(\tilde{X})$  with  $\tilde{1}(\tilde{x}) = 1$  for all  $\tilde{x} \in \tilde{X}$ . Then

$$\begin{aligned} (1 \circ \Pi)(\tilde{x}) &= 1(\Pi(\tilde{x})) \\ &= 1. \end{aligned}$$

Thus  $1 \circ \Pi = \tilde{1}$ . Hence  $\Pi^*(1) = 1$ .

**Definition 2.1.** Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $F$  be a subset of  $A$ . The *root extension* of  $A$ , denoted by  $\tilde{A}$ , is the smallest closed subalgebra of  $C(\tilde{X})$  containing  $\Pi^*(A)$  and the function  $p_f$  ( $f \in F$ ). The algebra  $\tilde{A}$  is called the  $F$ -*extension* of  $A$ .

We state and prove some results concerning  $\tilde{A}$ .

**Theorem 2.7.** Let  $A$  be a uniform algebra on a compact space  $X$ . Then

$\tilde{A}$  is a uniform algebra on a compact space  $\tilde{X}$ .

**Proof.** Since  $\Pi^*(1) = 1$  (lemma 2.6), so  $\tilde{A}$  contains the constants functions. We shall show that  $\tilde{A}$  separates the points of  $\tilde{X}$ .

Let  $\tilde{x}_i$  ( $i=1,2$ ) be distinct elements of  $\tilde{X}$  such that  $\tilde{x}_i = (x_i, z_i)$ .

We shall find a function  $g \in \tilde{A}$  with  $g(\tilde{x}_1) \neq g(\tilde{x}_2)$ .

Firstly, suppose that  $x_1 = x_2$ . Then there exists  $f \in A$  with  $f(x_1) \neq f(x_2)$ . Set

$$g = \Pi^*(f) \quad (f \in A).$$



$$\begin{aligned}
 g(\tilde{x}_1) &= (f \circ \Pi)(\tilde{x}_1) \\
 &= f(\Pi(\tilde{x}_1)) \\
 &= f(x_1) \\
 &\neq f(x_2) \\
 &= g(\tilde{x}_2).
 \end{aligned}$$

Secondly, suppose that the coordinate functionals  $Z_1, Z_2 \in C^F$  with  $Z_1 \neq Z_2$ . Then there exists  $f \in F$  with  $Z_1(f) \neq Z_2(f)$ . Take  $g = p_f$ . This completes the proof.

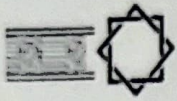
**Lemma 2.8.** Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $\tilde{A}$  be the  $F$ -extension of  $A$ . Let  $f, g \in A$ . Then

$$\Pi^*(fg) = \Pi^*(f) \Pi^*(g).$$

**Proof.** Let  $\tilde{x} \in \tilde{X}$ . Then

$$\begin{aligned}
 (\Pi^*(fg))(\tilde{x}) &= ((fg) \circ \Pi)(\tilde{x}) \\
 &= (fg)(\Pi(\tilde{x})) \\
 &= f(\Pi(\tilde{x})) g(\Pi(\tilde{x})) \\
 &= (\Pi^*f)(\tilde{x}) (\Pi^*g)(\tilde{x}).
 \end{aligned}$$

This completes the proof.



**Lemma 2.9.** Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $\tilde{A}$  be the  $F$ -extension of  $A$ . Let  $f \in A$ . If  $f$  is invertible, then  $\Pi^*(f)$  is invertible.

**Proof.** Suppose  $f$  is an invertible element in  $A$ . Then there exists  $g \in A$  such that  $fg = 1$ . We have

$$\begin{aligned}\Pi^*(f) \Pi^*(g) &= \Pi^*(fg) \\ &= \Pi^*(1) \\ &= 1.\end{aligned}$$

Hence  $\Pi^*(f)$  is invertible.

**Lemma 2.10.** Let  $A$  be a uniform algebra on a compact space  $X$ . Let  $\tilde{A}$  be the  $F$ -extension of  $A$ . Let  $f \in A$ . If  $f$  is invertible, then  $p_f$  is invertible.

**Proof.** Let  $f \in A$  be an invertible element. Then  $\Pi^*(f)$  is invertible (Lemma 2.9) Since  $\Pi^*(f) = p_f^2$ , so by Lemma 1.6,  $p_f$  is invertible.

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