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فضاءات للدوال التفاضلية
الغيـــر محــدودة

Spaces of infinitely differentiable functions

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فضاءات للدوال التفاضلية الغير محدودة

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الخلاصة

فى هذا البحث سوف نعطى نوعين من المتتبعات للاعداد الموجبة $(M_n)_{n=0}^{\infty}$ و التى تسمى المتتابعات الجبرية و المتتابعات اللوغاريتيمة المحدبة. و ايضا سوف نقدم نوعين من الفضاءات للدوال التفاضلية الغير محدودة $C(M_n)$, $E(2, M_n)$ و المرتبطة بالمتتابعات السابقة.سوف نقدم بعض النتائج و الخواص المتعلقة بهذه الفضاءات.

Abstract

In this paper, we give two sequences of positive numbers $(M_n)_{n=0}^{\infty}$ which are algebra sequences and logarithmically convex sequences. Also, we give two spaces of infinitely differentiable functions $C(M_n)$ and $E(2, M_n)$ which are related to the previous sequences. We give some properties and results concerning these spaces.

1. Introduction

In this section, we shall give some standard definitions and results which we shall use later in this paper.

Notation. Let $\mathbb{R}$ denote the set of all real numbers and let $\mathbb{C}$ denote the set of all complex numbers.

Definition 1.1. Let X be a non-empty set, and let f be a bounded complex-valued function on X. The *uniform norm* of f on X, denoted by $\| f \|_X$, is defined by

$$\| f \|_X = \sup \{ | f(x) | : x \in X \}.$$

Definition 1.2. A *normed algebra* is an algebra A which is normed as a linear space and in which

$$|| ab || \leq || a || \, || b || \quad (a,b \in A).$$

We assume $\| 1 \| = 1$.

Definition 1.3. A complete normed algebra which is a Banach space is called a *Banach algebra.*

There are two kinds of derivatives of complex functions.

Notation. We denote K' by the set of all limits points.

Definition 1.4. Let K be a compact subset of $\mathbb{C}$. Then K is called a *perfect* if $K = K'$.

Definition 1.5. Let K be a perfect, compact subset of $\mathbb{C}$. We say that $f : K \to \mathbb{C}$ is *complex- differentiable* at a point $a \in K$ if the limit

$$f'(a) = \lim_{z \to a, z \in K} \frac{f(z) - f(a)}{z - a} \quad \to (1)$$

exists. We say that f is *differentiable* on K if it is differentiable at each point of K. Definition (1) is equivalent to :

A function $f : K \to C$ is called *complex- differentiable* at a point $a \in K$ if for each $\in > 0$ there exists $\delta > 0$ such that

$$\left| \frac{f(z) - f(a)}{z - a} - f'(a) \right| < \in$$

for all $z \in K$ and $0 < | z - a | < \delta$.

It follows that $| f(z) - f(a) - f'(a)(z - a) | < \in | z - a |$.

Notation. Let K be a perfect, compact subset of C. We denote the n-th complex derivative of f at a point $a \in K$ by $f^{(n)}(a)$.

Definition 1.6. Let K be a perfect, compact subset of Then f is called *infinitely complex-differentiable* on K if $f^{(n)}$ exist on K for all $n \geq 1$.

Definition 1.7. Let K be a perfect, compact subset of C. Let $f : K \to C$ be a complex- differentiable function at a point $a \in K$. Let T be a linear map from $\mathbb{C}$ into $\mathbb{C}$ and for each $\in > 0$ there exists $\delta > 0$ such that

$$| f(z) - f(a) - T(z - a) | < \in | z - a | \quad \to (2)$$

for all $z \in K$ and $0 < | z - a | < \delta$.

The unique linear transformation which satisfies (2) is called the *Frechet derivative* of f at a and is denoted by $T(z - a) = D(f)(a)$.

Notation. We denote the n-th Frechet derivative of a function $f:\mathbb{R}^n \to \mathbb{C}$ by $D^n(f)$.

Let $\mathbb{R}^n\ (n \geq 1)$ be the Euclidean space. If $n = 1$, then a complex derivative and a Frechet derivative are the same. Let $n = 2$. If a complex derivative $f'(a)$ and the Frechet derivative $D(f)(a)$ both exist, then for $x \in \mathbb{C} = \mathbb{R}^2$, we have

$$D(f)(a)(x) = f'(a)x,$$

and so

$$\left|D^n(f)(a)\right| = \left|f^{(n)}(a)\right|.$$

Lemma 1.1. [2]. *Let* $D(f)$ *be the Frechet derivative of* f. *Let* $\alpha \in \mathbb{C}$. *Then*

(i) $D(\alpha f) = \alpha D(f)$.

(ii) $D^n(\alpha f) = \alpha D^n(f)\ (n \in \mathbb{N})$.

Notation. Let $C^\infty(\mathbb{R})$ denote the class of all differentiable complex functions on $\mathbb{R}$.

Theorem 1.1 [1] . *Let* $f, g \in C^\infty(\mathbb{R})$. *Then*

(i) $f + g \in C^\infty(\mathbb{R})$

(ii) $fg \in C^\infty(\mathbb{R})$.

(iii) $\alpha f \in C^\infty(\mathbb{R})$ (α is constant).

Theorem 1.2 (Leibnitz's rule) [3]. *Let $f , g \in C^{\infty}(\mathbb{R})$ and $a \in \mathbb{R}$. Let $n \in \mathbb{N}$. Then*

$$(f \cdot g)^{(n)}(a) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(a) . \, g^{(n-j)}(a)$$

Remark. Let $f , g \in C^{\infty}(\mathbb{R})$ and $a \in \mathbb{R}$. Leibnitz's rule gives us

$$\left|(f \cdot g)^{(n)}(a)\right| \leq \sum_{j=0}^{n} \binom{n}{j} \left|f^{(j)}(a)\right| \cdot \left|g^{(n-j)}(a)\right| .$$

2. Algebra and logarithmically convex sequences

In this section, we introduce two kinds of sequences of positive real numbers.

Definition 2.1 [3]. Let (M_n) be a sequence of positive real numbers . If the sequence (M_n) satisfies $M_0 = M_1 = 1$ and

$$\frac{M_n}{M_k \, M_{n-k}} \geq \binom{n}{k} ,$$

for all non-negative integers k, n with $k \leq n$, then we say that (M_n) is an *algebra sequence* .

Example 2.1. Let $M_n = (n!)^2$ $(n \in \mathbb{N})$. Then

$$\frac{M_n}{M_k M_{n-k}} = \frac{(n!)^2}{(k!)^2 ((n-k)!)^2}$$

$$= \left(\frac{n!}{k!\,(n-k)!} \right)^2$$

$$= \binom{n}{k}^2$$

$$\geq \binom{n}{k}.$$

Hence (M_n) is an algebra sequence.

In the same way, we can show that the sequence $(n!)^\alpha$ $(\alpha \geq 1)$ is an algebra sequence.

Lemma 2.1 *If* (M_n) *is an algebra sequence, then* $M_n \geq n!$ *for all n.*

Proof. Let (M_n) be an algebra sequence. Then

$$\frac{M_n}{M_k M_{n-k}} \geq \binom{n}{k},$$

for all $k \leq n$

Let $k = 1$. Then

$$\frac{M_n}{M_1 M_{n-1}} \geq \binom{n}{1} = n.$$

So

$$\frac{M_n}{1 . M_{n-1}} \geq n .$$

Hence $M_n \geq n M_{n-1}$. It follows that

$$M_n \geq n M_{n-1} \geq n (n-1) M_{n-2} \geq \ldots \geq n! M_0 .$$

Thus $M_n \geq n!$.

Definition 2.2. [4] Let (M_n) be a sequence of positive real numbers such that $M_0 = 1$ and

$$M_n^2 \leq M_{n-1} M_{n+1} \quad (n \geq 1).$$

Then (M_n) is called a *logarithmically convex* sequence.

Example 2.2. Let $M_n = n! \quad (n \in \mathbb{N})$. Then

$$\begin{aligned} M_{n+1} M_{n-1} &= (n+1)!(n-1)! \\ &= (n+1) n!(n-1)! \\ &= (n+1) n(n-1)!(n-1)! \\ &= (n+1) n((n-1)!)^2 \\ &> (n!)^2 . \end{aligned}$$

Hence (M_n) is a logarithmically convex sequence.

Lemma 2.2 *If* (M_n) *is a logarithmically convex sequence, then*

$$M_j \ M_{n-j} \leq M_n \quad (j \leq n)$$

Proof. Set $A_n = \log(M_n)$. Since (M_n) is logaritmically convex sequence, so

$$2 A_n \leq A_{n-1} + A_{n+1} .$$

So

$$A_n \leq \frac{A_{n-1} + A_{n+1}}{2} .$$

Let $j = 1,2,\ldots, n-j,\ldots,n$. The convexity of $\log(M_n)$ shows that

$$\frac{A_j - A_0}{j} \leq \frac{A_n - A_{n-j}}{j} .$$

It follows that

$$A_j - A_0 \leq A_n - A_{n-j} .$$

Since $A_0 = 0$, so we have

$$A_j + A_{n-j} \leq A_n \quad .$$

Hence $\log(M_j) + \log(M_{n-j}) \leq \log(M_n)$

$$\log(M_j \ M_{n-j}) \leq \log(M_n) .$$

It follows that $M_j \ M_{n-j} \leq M_n \quad (j \leq n)$.

Lemma 2.3. *If* (M_n) *is a logaritmically convex sequence, then*

$$(\text{i})\quad M_{j+1}\, M_n \le M_j\, M_{n+1} \quad (j \le n).$$

$$(\text{ii})\quad M_n\, M_m \le M_{n+m} \quad \text{for all } m, n.$$

3. Spaces of infinitely differentiable functions

In this section, we introduce two kinds of spaces of infinitely differentiable functions. Also, we give some their properties in some results.

Definition 3.1. Let (M_n) be a sequence of positive real numbers. We define the space $C(M_n)$ as follows :

$$C(M_n) = \left\{ f \in C^{\infty}(\mathbb{R}) : \left\| f^{(n)} \right\|_{\mathbb{R}} \le \mu_f\, \gamma_f^n\, M_n \ (n = 0, 1, 2, \ldots, n) \right\},$$

where $f^{(n)}$ is the n-th derivative of f , the norm

$$\| f \|_{\mathbb{R}} = \sup \{ f(x) : -\infty < x < \infty \}$$

is the uniform norm over $\mathbb{R}$ and μ_f , γ_f are positive constants depending on f .

Lemma 3.1. Let $f, g \in C(M_n)$. Then $f + g \in C(M_n)$.

Proof. Let $f, g \in C(M_n)$. Then

$$\left\| f^{(n)} \right\|_{\mathbb{R}} \le \mu_f\, \gamma_f^n\, M_n$$

$$\| g^{(n)} \|_{\mathbb{R}} \leq \mu_g \gamma_g^n M_n .$$

Thus $\quad \| (f+g)^{(n)} \|_{\mathbb{R}} = \| f^{(n)} + g^{(n)} \|_{\mathbb{R}}$

$$\leq \| f^{(n)} \|_{\mathbb{R}} + \| g^{(n)} \|_{\mathbb{R}}$$

$$\leq \mu_f \gamma_f^n M_n + \mu_g \gamma_g^n M_n$$

$$\leq (\mu_f \gamma_f^n + \mu_g \gamma_g^n) M_n .$$

Hence $f+g \in C(M_n)$.

Lemma 3.2. Let α be a constant. Let $f \in C(M_n)$. Then $\alpha f \in C(M_n)$.

Proof. Let $f \in C(M_n)$. Then $\| f^{(n)} \|_{\mathbb{R}} \leq \mu_f \gamma_f^n M_n$. So

$$\| (\alpha f)^{(n)} \|_{\mathbb{R}} \leq \mu_{\alpha f} \gamma_{\alpha f}^n M_n .$$

Hence $\alpha f \in C(M_n)$.

Lemma 3.3. Let $f, g \in C(M_n)$. Let α, β be constants. Then

$$\alpha f + \beta g \in C(M_n).$$

Proof. The proof follows by Lemma 3.1 and Lemma 3.2.

Lemma 3.4. Let α be a constant. Then $C(M_n) = \alpha\, C(M_n)$.

Proof. Let $f \in C(M_n)$. Then $\| f^{(n)} \|_{\mathbb{R}} \le \mu_f \gamma_f^n M_n$.

Set $\mu'_f = \frac{1}{\alpha} \mu_f$. Then

$$\| f^{(n)} \|_{\mathbb{R}} \le \alpha \mu'_f \gamma_f^n M_n$$

$$\le \mu'_f \gamma_f^n (\alpha M_n).$$

Thus $f \in C(\alpha M_n)$. Now, let $f \in C(\alpha M_n)$. Then

$$\| f^{(n)} \|_{\mathbb{R}} \le \mu_f \gamma_f^n (\alpha M_n)$$

$$= (\alpha \mu_f) \gamma_f^n M_n,$$

where $\alpha \mu_f$ is a new constant. So $f \in C(M_n)$.

It follows that $C(M_n) = \alpha C(M_n)$.

Lemma 3.5. Let $f \in C(M_n)$. Then

$$\limsup_{n \to \infty} \left(\frac{\| f^{(n)} \|_{\mathbb{R}}}{M_n} \right)^{\frac{1}{n}} \le \gamma_f .$$

Proof. Let $f \in C(M_n)$. Then

$$\limsup_{n\to\infty}\left(\frac{\|f^{(n)}\|_{\mathbb{R}}}{M_n}\right)^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left(\frac{\mu_f\,\gamma_f^n\,M_n}{M_n}\right)^{\frac{1}{n}}$$

$$= \limsup_{n\to\infty}\left(\mu_f\,\gamma_f^n\right)^{\frac{1}{n}}$$

$$= \gamma_f \limsup_{n\to\infty}\left(\mu_f\right)^{\frac{1}{n}}.$$

Since μ_f is a positive, so $\limsup_{n\to\infty}\left(\mu_f\right)^{\frac{1}{n}} = 1.$

Hence $\limsup_{n\to\infty}\left(\frac{\|f^{(n)}\|_{\mathbb{R}}}{M_n}\right)^{\frac{1}{n}} \leq \gamma_f.$

Lemma 3.6. *Let* (M_n) *be a logaritmically convex sequence. Let* $f, g \in C(M_n)$. *Then* $f\,g \in C(M_n)$.

Proof. Let $f, g \in C(M_n)$. Then

$$\|f^{(n)}\|_{\mathbb{R}} \leq \mu_f\,\gamma_f^n\,M_n$$

$$\|g^{(n)}\|_{\mathbb{R}} \leq \mu_g\,\gamma_g^n\,M_n.$$

Leibnitz's rule gives us

$$\| (f g)^{(n)} \|_{\mathbb{R}} \leq \mu_f \mu_g \sum_{j=0}^{n} \binom{n}{j} \gamma_f^j \gamma_g^{n-j} M_j M_{n-j}$$

$$\leq \mu_f \mu_g (\gamma_f + \gamma_g)^n M_n .$$

Thus $f g \in C(M_n)$.

Definition 3.2. Let (M_n) be a sequence of positive numbers. We define the space $E(2, M_n)$ as follows :

$$E(2,M_n) = \{ f \in C^{\infty}(\mathbb{R}^2) : \sum_{n=0}^{\infty} \left(M_n^{-1} \| D^n(f) \|_{\mathbb{C}} \right) < \infty \},$$

where $\| D^n(f) \|_{\mathbb{C}} = \sup \left\{ \| D^n(f)(a) \| : a \in \mathbb{C} \right\}$.

We give some properties of $E(2,M_n)$ in the next results. The proofs are the same of proof of results of the space $C(M_n)$.

Lemma 3.7. *Let $f, g \in E(2,M)$. Then $f + g \in E(2,M)$.*

Lemma 3.8. *Let α be a constant. Let $f \in E(2,M_n))$. Then $\alpha f \in E(2,M_n)$.*

Lemma 3.9. *Let M be an algebra sequence. Let $f, g \in E(2,M_n)$. Then $f \cdot g \in E(2,M_n)$.*

Theorem 3.1. [4] . *Let M be a sequence of positive numbers. Then $E(2,M_n))$ is a Banach space with respect to the norm*

$$\|f\| = \sum_{n=0}^{\infty} M_n^{-1} \|D^n(f)\|_{\mathbb{C}} \quad (f \in E(2, M_n)).$$

Theorem 3.2. *Let* M *be an algebra sequence. Then* $E(2, M_n)$ *is a Banach algebra.*

Proof. By Theorem 3.1, $E(2, M_n)$ is a Banach space. Let $f, g \in E(2, M_n)$. Then

$$\sum_{n=0}^{\infty} M_n^{-1} \|D^n(fg)\|_{\mathbb{C}} \leq \sum_{n=0}^{\infty} M_n^{-1} \left\|\sum_{j=0}^{n} \binom{n}{j} D^{(j)}(f).D^{(n-j)}(g)\right\|_{\mathbb{C}}$$

$$\leq \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{M_j} \|D^{(j)}(f)\|_{\mathbb{C}} \frac{1}{M_{n-j}} \|D^{(n-j)}(g)\|_{\mathbb{C}}$$

$$\leq \|f\| \|g\|$$

Thus $\|fg\| \leq \|f\| \|g\|$.

That is $E(2, M_n)$ is a normed algebra.

Hence $E(2, M_n)$ is a Banach algebra.

Notation.. Let f be a complex-valued function distribution on $\mathbb{C}$.

Set

$$\bar{\partial}(f)=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\,\frac{\partial f}{\partial y}\right).$$

We shall need the following lemma in the next theorem..

Lemma 3.10. [4]. *Let f be a complex differentiable function. Then*

$$\left\|D^{n}(\bar{\partial}(f))\right\|_{\mathbb{C}}\leq\left\|D^{n+1}(f)\right\|_{\mathbb{C}}.$$

Definition 3.3. Let (M_n) be a sequence of positive numbers. We define a new sequence of positive numbers (M_n^-) as follows

$$M_n^-=\begin{cases}M_{n-1} & ,n>0\\ 1 & ,n=0.\end{cases}$$

Theorem 3.4. *Let $f\in E(2,M_n)$. Then*

$$\left\|\bar{\partial}(f)\right\|_{E(2,\,M_n)}\leq\left\|f\right\|_{E(2,\,M_n^-)}.$$

Proof. *Let* $f\in E(2,M_n)$. We have

$$\left\|\bar{\partial}(f)\right\|_{E(2,\,M_n)}=\sum_{n=0}^{\infty}M_n^{-1}\left\|D^{n}(\bar{\partial}(f))\right\|_{\mathbb{C}}$$

$$\leq \sum_{n=0}^{\infty} M_n^{-1} \left\| D^{n+1}(f) \right\|_{\mathbb{C}} \quad (\text{ Lemma 3.10 }).$$

We have

$$\| f \|_{E(2, M_n^-)} = \sum_{n=0}^{\infty} \left(M_n^-\right)^{-1} \left\| D^n(f) \right\|_{\mathbb{C}}$$

$$= \left(M_0^-\right)^{-1} \left(\| f \|_{\mathbb{C}} \right) + \sum_{n=1}^{\infty} \left(M_n^-\right)^{-1} \left\| D^n(f) \right\|_{\mathbb{C}}$$

$$= \| f \|_{\mathbb{C}} + \sum_{n=1}^{\infty} \left(M_n^-\right)^{-1} \left\| D^n(f) \right\|_{\mathbb{C}}$$

$$= \| f \|_{\mathbb{C}} + \sum_{n=1}^{\infty} \left(M_{n-1}\right)^{-1} \left\| D^n(f)) \right\|_{\mathbb{C}} .$$

Hence $\| f \|_{E(2, M_n^-)} \geq \sum_{n=1}^{\infty} \left(M_{n-1}\right)^{-1} \left\| D^n(f) \right\|_{\mathbb{C}}$

$$= \sum_{n=0}^{\infty} M_n^{-1} \left\| D^{n+1}(f)) \right\|_{\mathbb{C}} .$$

Thus $\left\| \bar{\partial}(f) \right\|_{E(2, M_n)} \leq \| f \|_{E(2, M_n^-)}$.

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