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A Result on Fixed Points

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ABSTRACT

Let T be a function from a non-empty set X into itself such that $Tx = x$. Then x in X is a fixed point of T. In this paper, we give the generalized of Sehgal Theorem in the area of uniqueness of fixed points in the case of metric spaces.

KeyWords : Fixed points, Cluster points, metric spaces.

Before, we study the next theorem, we state the following lemma which help us to proof it.

Lemma 1. *Let T be a mapping from a non-empty set X into itself. If T has a fixed point x in X, then T^n, $(n > 1)$ has the same fixed point x.*

Theorem 2. *Let (X, d) be a metric space. Let T be a continuous mapping from (X, d) into itself satisfies*

$$d(Tx, Ty) < c \max\{d(x, Tx), d(y, Ty), d(x, y)\},$$

for all $x, y \in X$ with $x \neq y$ and $0 < c < 1$. *Suppose the sequence $\{T^n z\}$ has a cluster point u for some z in X. Then the sequence $\{T^n z\}$ converges to u and u is the unique fixed point of T.*

Proof.

We have two cases.

Case (i): Let $T^{n+1} z = T^n z$ for some non-negative integers n.

Then

$$T(T^n z) = T^{n+1} z = T^n z .$$

Hence $T^n z$ is a fixed point of T. Set $T^n z = u$ ($u \in X$). So $T u = u$. Thus u is a fixed point of T.

For uniqueness, let $T u = u$ and $T v = v$ such that $u \neq v$ Then

$$d(u,v) = d(T u, T v)$$

$$< c \max\{d(u,Tu), d(v,Tv), d(u,v)\}$$

$$= c\, d(u,v),$$

which is impossible. Clearly $\lim_{n\to\infty} T^n z = u$.

Case (ii): Let $T^{n+1} z \neq T^n z$ for all non-negative integers n.

Define a function U from X into $\mathbb{R}$ by

$$U(t) = d(t, T t) \quad \text{for all } t \in X .$$

Then U is continuous. We have

$$U(T^n z) = d(T^n z, T^{n+1} z)$$

$$= d(T(T^{n-1} z), T(T^n z))$$

$$< c \max\{d(T^{n-1} z, T^n z), d(T^n z, T^{n+1} z), d(T^{n-1} z, T^n z)\}$$

$$= c \max\{d(T^{n-1} z, T^n z), d(T^n z, T^{n+1} z)\}$$

$$= c \max\{U(T^{n-1} z), U(T^n z)\}.$$

Therefore

$$U(T^n z) < c \max\{U(T^{n-1} z), U(T^n z)\}.$$

If $U(T^n z) \geq U(T^{n-1} z)$, then $U(T^n z) < c\, U(T^n z)$ which is impossible.

Hence $U(T^n z) < c\, U(T^{n-1} z)$.

Thus $U(T^n z) < c\,U(T^{n-1} z) < c\,U(z)$ for all n.

It follows that $U(T^n z) < c\,U(T^{n-1} z) < c\,U(z) < U(z)$.

Thus the sequence $\{U(T^n z)\}$ is bounded below and increasing. So it is convergent. Let $\lim_{n \to \infty} U(T^n z) = p$. Then any subsequence $\{U(T^{n_i} z)\}$ of the sequence $\{U(T^n z)\}$ converges to the same limit p. Then

$$\begin{aligned} p &= \lim_{n \to \infty} U(T^n z) \\ &= \lim_{i \to \infty} U(T^{n_i} z) \\ &= U(\lim_{i \to \infty} T^{n_i} z) \\ &= U(u). \end{aligned}$$

So

$$\begin{aligned} U(Tu) &= U(T(\lim_{i \to \infty} T^{n_i} z)) \\ &= U(\lim_{i \to \infty} T^{n_i + 1} z) \\ &= p. \end{aligned}$$

Hence $U(u) = U(Tu) = p$ → (1)

On contrary, we assume that $Tu \neq u$. Then

$$\begin{aligned} U(Tu) &= d(Tu, T^2 u) \\ &< c \max\{d(u, Tu), d(Tu, T^2 u), d(u, Tu)\} \\ &= c \max\{d(u, Tu), d(Tu, T^2 u)\} \\ &= c \max\{U(u), U(Tu)\}. \end{aligned}$$

Therefore

$$U(Tu) < c \max\{U(u), U(Tu)\}.$$

If $U(u) \leq U(Tu)$, then $U(Tu) < c\,U(Tu)$ which is impossible. Then

$U(u) > U(Tu)$ contradicts to (1).

Thus $Tu = u$. We obtain

$$\begin{aligned} p &= \lim_{n\to\infty} U(T^n z) \\ &= \lim_{n\to\infty} d(T^n z, T^{n+1} z) \\ &= \lim_{i\to\infty} d(T^{n_i} z, T^{n_i+1} z) \\ &= d(\lim_{i\to\infty} T^{n_i} z, T(\lim_{i\to\infty} T^{n_i} z)) \\ &= d(u, Tu) \\ &= d(u,u) \\ &= 0. \end{aligned}$$

Hence $U(u) = U(Tu) = 0$.

Let $\in' > 0$. There exists a positive integer N such that $U(T^k u) < \in'$ for all $k > N$. Since the sequence $\{T^n z\}$ has a cluster point u for some z in X, so for $\in' > 0$, there exists a positive integer N such that $d(T^k u, u) < \in'$ for all $k > N$. Thus

$$\max\{U(T^k u), d(T^k u, u)\} < \in'.$$

Since $0 < c < 1$. So

$$c \max\{U(T^k u), d(T^k u, u)\} < c \in'.$$

We choose $\in = c \in'$. Thus

$$c \max\{U(T^k u), d(T^k u, u)\} < \in.$$

We have

$d(T^n z, T^n u)$

$$< c \max\{ d(T^{n-1} z, T^n z), d(T^{n-1} u, T^n u), d(T^{n-1} z, T^{n-1} u)\}$$

$$= c \max \{ U(T^{n-1} z), d(T^{n-1} z, T^{n-1} u) \}$$

$$\leq c \max \{ U(T^{n-1} z), \max \{ U(T^{n-2} z), d(T^{n-2} z, T^{n-2} u) \} \}.$$

So

$$d(T^{n} z, T^{n} u) = c \max \{ U(T^{n-2} z), d(T^{n-2} z, T^{n-2} u) \}.$$

In the same way, we can obtain

$$d(T^{n} z, T^{n} u) < c \max \{ U(T^{k} z), d(T^{k} z, u) \}$$

$$< \in.$$

Since $T^{n} u = u$ (Lemma 1), so $d(T^{n} z, u) < \in$.

This completes the proof.

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