# HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS ASSOIATED WITH GENERALIZED SRIVASTAVA–ATTIYA OPERATOR

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## Abstract

This paper is to introduce a certain class of analytic functions denoted by $\mathcal{R}_b^s(\alpha)$ which is defined by generalized Srivastava – Attiya operator. This operator is associated with Hurwitz-Lerch Zeta function, obtain an upper bound to the second Hankel determinant $|a_2\, a_4 - a_3^2|$ for the class $\mathcal{R}_b^s(\alpha)$.

**Keywords:** Hankel determinant; univalent functions; integral operator; Hurwitz-Lerch Zeta function.

## Introduction

Let $\mathcal{A}$ denotes the class of all analytic functions in the open unit disc

$$\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\},$$

and given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k . \qquad (1)$$

Let $S$ denotes the subclasses of $\mathcal{A}$ consisting of univalent functions.

For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) of $f$ and $g$ given by the power series

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k .$$

**Definition 1:** (Noonan and Thomas,1976) For the function $f$ given by (1) for $q \geq 1$ and $n \geq 1$, the qth Hankel determinant of $f$ is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors. For example, Noor (1993) determined the rate of growth of $Hq(n)$ as $n \to \infty$ for the functions in $\mathcal{S}$ with a bounded boundary. Ehrenborg (2000) studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman (2001) . Note that the Hankel determinant $H_2(1) = |a_3 - a_2^2|$ is related to the well-known classical theorem of Fekete-Szego. It is also known that Fekete-Szego gave sharp estimates of $|a_3 - \mu a_2^2|$ for $\mu$ real and $f \in S$. Many authors such as Darus ,2002, Darus and Hong 2004 studied the estimation of $|a_3 - \mu a_2^2|$ for various subclasses. In this paper, we consider the second Hankel determinant in the case of $q = 2$ and $n = 2$, namely,

$$\boldsymbol{H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.}$$

Janteng et al (2007) have considered the functional $H_2(2)$ and found a sharp bound for the function $f$ in the subclass $RT$ of $S$ , consisting of functions whose derivative has a positive real part defined as $\mathcal{R}\left\{f'(z)\right\} \geq 0$. In their work, they have shown that $f \in R$ then $H_2(2) \leq \frac{4}{9}$. In a previous work Janteng et al,2006 obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex functions denoted by $ST$ and $CV$ of $S$ . Also they have shown that $H_2(2) \leq 1$, and $H_2(2) \leq \frac{1}{8}$ respectively.

Based on the above-mentioned results obtained by other researchers, we determine the upper bounds of the second Hankel determinant $H_2(2)$) for functions belonging to subclass $\mathcal{R}_b^s(\alpha)$.

Nagat and Darus (2011,2012) introduce a general integral operator $\Im_{s,b}^{\alpha} f(z)$ . It has been defined by the means of the general Hurwitz Lerch Zeta function i.e defined on the class of normalized analytic functions in the open unit disc by using the similar approach Srivastava and Attiya operator (2007). This operator has been introduced by many researchers namely Owa , Srivastava Alexander and many others.

As we will show in the following:

**Definition 2:** Srivastava and Choi (2001) defined the general Hurwitz–Lerch Zeta function $\Phi(z,s,b)$ by

$$\Phi(z,s,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

where $(s \in \mathbb{C},\ b \in \mathbb{C} - \mathbb{Z}_0^-)$ *when* $(|z| < 1)$, and $(\Re(b) > 1)$ *when* $(|z| = 1)$.

Note that:

$$\Phi^*(z,s,b) = (b^s z\,\Phi(z,s,b)) * f(z) = z + \sum_{k=2}^{\infty} \frac{b^s}{(k+b-1)^s} a_k z^k.$$

Owa and Srivastava (1987 ) introduced the operator $\Omega^{\alpha}:\mathcal{A}\to\mathcal{A},$ . It is known as an extension of the fractional derivative and the fractional integral as follows:

$$\Omega^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}a_k z^k \qquad (\alpha\neq 2,3,4,\cdots),$$

where $D_z^{\alpha}f(z)$ the fractional derivative of $f$ of order $\alpha$ see (Owa and Srivastava,1984). For $s\in\mathbb{C}$, $b\in\mathbb{C}-\mathbb{Z}_0^-$, and $0\leq\alpha<1,$ the generalized integral operator

$(\Im_{s,b}^{\alpha}f):\mathcal{A}\to\mathcal{A}$ is defined by

$$\Im_{s,b}^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}\Phi^{*}(z,s,b), \quad (\alpha\neq 2,3,4,\cdots)$$

$$=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\left(\frac{b}{k-1+b}\right)^{s}a_k z^k, \quad (z\in\mathbb{U}). \quad (2)$$

Note that : $\Im_{0,b}^{0}f(z)=f(z).$

Special cases of this operator include:

- $\Im_{0,b}^{\alpha}f(z)\equiv\Omega^{\alpha}f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}a_k z^k$ is Owa and Srivastava operator (1984).
- $\Im_{s,b+1}^{0}f(z)\equiv J_{s,b}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{b+1}{k+b}\right)^{s}a_k z^k$ is Srivastava and Attiya integral operator (2007).
- $\Im_{1,1}^{0}f(z)\equiv A(f)(z)=\int_0^z\frac{f(t)}{t}dt=z+\sum_{k=2}^{\infty}\frac{1}{k}a_k z^k,$ is Alexander integral operators (1915).
- $\Im_{1,2}^{0}f(z)\equiv L(f)(z)=\frac{2}{z}\int_0^z f(t)dt=z+\sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)a_k z^k,$ is Libera integral operators (1969).
- $\Im_{\sigma,2}^{0}f(z)\equiv I^{\sigma}f(z)=z+\sum_{k=2}^{\infty}\left(\frac{2}{k+1}\right)^{\sigma}a_k z^k,$ is Jung– Kim– Srivastava integral operator (1993).

Motivated by Janteng et al (2006), in the present paper, we seek the upper bound of the functional $|a_2a_4-a_3^2|$ for functions *f* which belong to the subclass $\mathcal{R}_b^s(\alpha)$ .The techniques used here follow the same with the one in ( Janteng et al,2006).

The subclass $\mathcal{R}_b^s(\alpha)$ is defined as the following.

**Definition 3:** The function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}_b^s(\alpha)$ if it satisfies the inequality

$$\mathcal{R}\left\{\left(\Im_{s,b}^{\alpha} f(z)\right)'\right\} > \mathbf{0}, \quad z \in \mathcal{U}, \qquad \textbf{(3).}$$

The subclass $\mathcal{R}_0^0(\alpha)$ was studied by MacGregor (1962) who referred to numerous earlier investigations that involved functions whose derivative has a positive real part.

**Preliminary Results**

To prove our main results, we need the following Lemmas.

Let $P$ be the family of all functions $p$ analytic in $\mathcal{U}$ for which $\mathcal{R}\{p\} > 0$ be given by the power series

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad z \in \mathcal{U}.$$

**Lemma 1:** (Duren, 1983) *If* $p \in P$ *then*

$$|c_k| \le 2, \quad for \quad all \quad k \in \mathbb{N}. \qquad (4)$$

**Lemma 2:** (Libera and Zlotkiewicz. 1982, 1983). Let the function $p \in P$ be given by the power series Then,

$$2c_2 = c_1^2 + x(4 - c_2), \qquad (5)$$

for some $x$, $|x| \le 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)2(1 - |x|^2)z, \qquad (6)$$

for some $z$, $|z| \le 1$.

**Main Results**

Our main result is the following.

**Theorem 1**

Let the function $f$ given by (1) be in the subclass. $\mathcal{R}_b^s(\alpha)$ then

$$\left|\boldsymbol{a_2 a_4 - a_3^2}\right| \le \frac{\boldsymbol{4(\Gamma(4-\alpha))^2(b+2)^{2S}}}{\boldsymbol{9(\Gamma(2-\alpha))^2(\Gamma(2-\alpha))^2}}.$$

The result obtained is sharp.

**Proof**

We refer to the method by (Libera and Zlotkiewicz, 1982, 1983). For it follows

from (3) that $\exists\, p \in P$ such that

$$\left(\Im_{s,b}^{\alpha} f(z)\right)' = \boldsymbol{p(z) = 1 + C_1 z + C_2 z^2 + C_3 z^3 + \cdots}, \qquad (\mathbf{7})$$

for some ($z \in \mathcal{U}$). From (7) computation and equating coefficients, we obtain

$$a_2 = \frac{C_1\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\,\Gamma(2-\alpha)b^s},$$

$$a_3 = \frac{C_2\Gamma(4-\alpha)(b+2)^s}{3\Gamma(4)\,\Gamma(2-\alpha)b^s} \qquad (8)$$

$$a_4 = \frac{C_3\Gamma(5-\alpha)(b+3)^s}{4\Gamma(5)\,\Gamma(2-\alpha)b^s}\ .$$

From (8), it can be easily established that

$$\left|a_2\, a_4 - a_3^2\right| = \frac{1}{(\Gamma(2-\alpha))^2 b^{2s}}\left|\frac{C_1C_3\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s}{8\Gamma(3)\,\Gamma(5)} - \frac{C_2^2(\Gamma(4-\alpha))^2(b+2)^{2s}}{9(\Gamma(4))^2}\right| \qquad (9)$$

Since the function $p(z)$ is a member of the class $P$ simultaneously, we assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c, (c \in [0,2])$.

Using (5) along with (6), we get

$$\left|a_2\, a_4 - a_3^2\right| = \frac{1}{(\Gamma(2-\alpha))^2 b^{2s}}\left|\frac{C_1[C_1^3+2xC_1(4-C_1^2)-C_1x^2(4-C_1^2)}{32\Gamma(3)\,\Gamma(5)} + \frac{2C_1(4-C_1^2)(1-|x|^2)z\Gamma(3-\alpha)\Gamma(5-\alpha)(b+1)^s(b+3)^s}{32\Gamma(3)\,\Gamma(5)} - \frac{\left[C_1^4+2xC_1^2(4-C_1^2)+x^2(4-C_1^2)^2\right](\Gamma(4-\alpha))^2(b+2)^{2s}}{36(\Gamma(4))^2}\right|, =$$

$$\frac{1}{(\Gamma(2-\alpha))^2 b^{2s}}\left|\frac{C_1^4+2xC_1^2(4-C_1^2)-C_1^2x^2(4-C_1^2)}{32\Gamma(3)\,\Gamma(5)} + \frac{2C_1(4-C_1^2)(1-|x|^2)z\Gamma(3-\alpha)\Gamma(5-\alpha)(b+1)^s(b+3)^s}{32\Gamma(3)\,\Gamma(5)} - \frac{\left[C_1^4+2xC_1^2(4-C_1^2)+x^2(4-C_1^2)^2\right](\Gamma(4-\alpha))^2(b+2)^{2s}}{36(\Gamma(4))^2}\right|,$$

$$, \quad \left|a_2\, a_4 - a_3^2\right| = \frac{1}{(\Gamma(2-\alpha))^2\,\Gamma(3)\,\Gamma(5)(\Gamma(4))^2\, b^{2s}}\left|\frac{C^4}{288}\left[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\left(\Gamma(4)\right)^2 - 8\Gamma(3)\,\Gamma(5)\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}\right] + \frac{C^2x(4-C^2)}{144}\left[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\left(\Gamma(4)\right)^2 - 8\Gamma(3)\,\Gamma(5)\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}\right] - \frac{x^2(4-C^2)}{288}\left[9C^2\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\left(\Gamma(4)\right)^2 - 8\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}(4-C^2)\right] + \frac{C(4-C^2)(1-|x|^2)\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{16}\right|,$$

$$\left|a_2\, a_4 - a_3^2\right| = \frac{1}{\left(\Gamma(2-\alpha)\right)^2\,\Gamma(3)\,\Gamma(5)\,\Gamma(4)^2\, b^{2s}}\left|\frac{C^4}{288}\left[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\left(\Gamma(4)\right)^2 - 8\Gamma(3)\,\Gamma(5)\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}\right] + \frac{C(4-C^2)}{16}\left[\Gamma(3-\alpha)\,\Gamma(5-\right.\right.$$

$$\alpha)(b+1)^s(b+3)^s\big(\Gamma(4)\big)^2\Big] + \frac{C^2x(4-C^2)}{144}\Big[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\big(\Gamma(4)\big)^2 - 8\Gamma(3)\,\Gamma(5)\,\big(\Gamma(4-\alpha)\big)^2(b+2)^{2s}\Big] - x^2(4-C^2)\Big[\frac{9C^2\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2 + 8\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}(4-C^2)}{288}\Big] - \frac{C[\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2]}{16}\Bigg|,$$

An application of triangle inequality and replacement of |x| by $\rho$ give

$$|a_2\,a_4 - a_3^2| \le \frac{1}{(\Gamma(2-\alpha))^2\,\Gamma(3)\,\Gamma(5)\,\Gamma(4)^2\,b^{2s}}\Big[\frac{C^4}{288}\big[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\big(\Gamma(4)\big)^2 - 8\Gamma(3)\,\Gamma(5)\,\big(\Gamma(4-\alpha)\big)^2(b+2)^{2s}\big] + \frac{C(4-C^2)}{16}\big[\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\big(\Gamma(4)\big)^2\big] + \big[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\big(\Gamma(4)\big)^2 - 8\Gamma(3)\,\Gamma(5)\,\big(\Gamma(4-\alpha)\big)^2(b+2)^{2s}\big]\frac{C^2\rho(4-C^2)}{144} + \Big[\frac{9C^2\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\Gamma(4)^2 + 8\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}(4-C^2)}{288} - \frac{C\big[\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2\big]}{16}\Big]\rho^2(4-C^2) = F(\rho), \qquad \textbf{(10).}$$

With $\rho = |x| \le 1$. we assume that the upper bound for (10) attains at the interior point $\rho \in [0,1]$ and $c \in [0, 2]$, then,

$$F'(\rho) = \frac{1}{(\Gamma(2-\alpha))^2\,\Gamma(3)\,\Gamma(5)\,(\Gamma(4))^2\,b^{2s}}\big[9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2 - 8\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}\big]\frac{C^2(4-C^2)}{144} + \Big[\frac{C^2\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{16} + \frac{\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}(4-C^2)}{18} + \frac{C\big[\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2\big]}{8}\Big]\rho(4-C^2),$$

$\frac{\Gamma(3)\,\Gamma(5)\,\Gamma(4-\alpha)^2(b+2)^{2s}}{\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s\Gamma(4)^2} \le \frac{9}{8}$, we observed that $F'(\rho) > 0$, for $\rho \in [0,1]$, implying that $F$ is an increasing function and thus the upper bound for (10) corresponds to $\rho = 1$ and so $maxF(\rho) = F(1)$. This contradicts our assumption of having the maximum value in the interior $\rho \in [0,1]$.

Now let

$$G(C) = F(1) = \frac{1}{(\Gamma(2-\alpha))^2\,\Gamma(3)\,\Gamma(5)(\Gamma(4))^2\,b^{2s}}\Bigg\{\frac{9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2C^4}{288} - \frac{8\Gamma(3)\,\Gamma(5)(\Gamma(4-\alpha))^2(b+2)^{2s}C^4}{288} + \frac{9\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2C^2\big(4-C^2\big)}{144} - \frac{8\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}C^2\big(4-C^2\big)}{144} + \frac{9C^2\big(4-C^2\big)\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{288} +$$

$$\frac{8\left(4-C^2\right)^2\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2(b+2)^{2s}}{288}-\frac{C\left(4-C^2\right)\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{288}+\frac{C\left(4-C^2\right)\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{288}\Bigg\}.$$

Assume that $G(c)$ has a maximum value in an interior of $c \in [0, 2]$, by elementary calculation we find

$$G'(C)=\frac{1}{(\Gamma(2-\alpha))^2\Gamma(3)\,\Gamma(5)(\Gamma(4))^2\,b^{2s}}\left\{\left[\frac{\Gamma(3-\alpha)\,\Gamma(5-\alpha)(b+1)^s(b+3)^s(\Gamma(4))^2}{4}-\frac{2\Gamma(3)\,\Gamma(5)\,(\Gamma(4-\alpha))^2}{9}\right]C(4-C^2)\right\},$$

so that $G'(c)<0$ *for* $(0<c<2)$, and has real critical point at $c=0$. Therefore, $\max G(c)$ for $(0<c<2)$, occurs at $c=0$. Therefore, the upper bound of corresponds to $c=0$ and $\rho=1$ .

Hence $$\left|a_2\,a_4-a_3^2\right|\ \le\frac{1}{(\Gamma(2-\alpha))^2\,\Gamma(4)^2\,b^{2s}}\left[\frac{4\,(\Gamma(4-\alpha))^2(b+2)^{2s}}{9}\right].$$

This concludes the proof of our theorem.

Taking $s=\alpha=0,$ in Theorem 1, reduces to the result of Janteng et al, (2006), Theorem 3.1, p.6.

## Conclusions

This paper estimates the coefficients for a general class of analytic functions, which is related to the Hankel determinant of second order. The researcher calls for studying the Hankel determinant of other classes of the analytic functions through using the same operator or generalized hypergeometric functions.

## References


- Nagat.M.Mustafa and Maslina Darus, (2012), Inclusion relations for subclasses of analytic functions defined by integral operator associated Hurwitz- Lerch Zeta function, Tansui Oxfored journal of Information and Mathematics Sciences 28(4): 379-393.

- Nagat.M.Mustafa and Maslina Darus,(2011), On a subclass of analytic functions with negative coefficient associated to an integral operator involving Hurwitz- Lerch Zeta function, ”Vasile Alecsandri” University of Bacau Faculty of Sciences Scientific Studies and Research Series Mathematics and Informatics.21(2): 45 – 56.

- J.W. Noonan and D. K. Thomas, (1976), On the second Hankel determinant of a really mean p-valent functions, Trans. Amer. Math. Soc, 223(2): 337–346.

- Noor K.(1993), Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl, 28(8) , 731-739.

- Ehrenborg R,(2000), The Hankel determinant of exponential polynomials, American Mathematical Monthly, 107 ,557-560.

- Layman J.W,(2001), The Hankel transform and some of its properties, J. of Integer Sequences, 4:1-11.

- T.H. MACG REGOR, (1962), Functions whose derivative has a positive real part Trans. Amer. Math. Soc, 104 532–537.

- M. Darus, (2002), The Fekete-Szego problems for close-to-convex functions of the class $Ksh(\alpha,\beta\ )$, Acta Mathematica Academia Paedagogicae Nyiregy- haziness, 18(1): 13-18.

- M. Darus, and N.C. Hong, (2004), The Fekete-Szego problem for strongly close-to-convex functions with Hadamard product, Far East Journal of Mathematical Sciences (FJMS), 15(1) .1-7.

- Janteng, S. A. Halim and M . Darus,(2007), Hankel determinant for starlike and convex functions, Int. J Math. Analysis, 1(13**):** 619-625.

- Janteng, S. Abdul Halim, and M. Darus, (2006), Coefficient inequality for a function whose derivative has a positive real part, Journal of Inequalities in Pure and Applied Mathematics, 7, 1-11.

- H. M. Srivastava and A. A. Attiya,(2007), An integral operator associated with the Hurwitz–Lerch Zeta function and differential subordination, Integral Transforms and Special Functions,18(.3): 207-216.

- R.J. Libera, (1969), Some classes of regular univalent functions, Proceedings of the American Mathematical Societ, 135,429-449.

- Srivastava, H.M. and Choi. J,(2001), Series Associated with the Zeta and Related Functions, Dordrecht Boston and London: Kluwer Academic Publishers.

- S. Owa and H. M. Srivastava,(1987), Univalent and starlike generalized hypergeometric functions, Canadian Journal of Mathematics, 39(5), 1057-1077.

- H. M. Srivastava and S. Owa,(1984) An application of the fractional derivative, Mathematica Japonica, 29(3), 383-389.

- Jung, I.B., Kim, Y.C. and Srivastava, H.M, (1993) The Hardy space of analytic functions associated with certain one-parameter families of integral operators, Journal of Mathematical Analysis and Applications, 176,138-147.

- J.W. Alexander,(1915), Functions which map the interior of the unit circle upon simple regions, Annals of Mathematics, 17, 12-22.

- P. L. Duren, (1983) "Univalent functions," in Grundlehren der Mathematischen Wissenschaften, vol. 259, Springer, New York, NY, *USA.*

- R. J. Libera and E. J. Zlotkiewicz, (1983), Coefficient bounds for the inverse of a function with derivative in P, Proc. Amer.Math. Soc., 87(2251–289.

- R. J. Libera and E.J. Zlotkiewicz, (1982), Early coefficients of the inverse of a regular convex function, Proc. Amer.Math. Soc., 85(2), 225-230.