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On Subclasses Of Uniformly Bazilevic Type Functions Using New Generalized Derivative Operator

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Abstract— In this work, the authors defined a certain classes of Bazilevic functions using generalized derivative operator. Having the analytic function, we discuss here some conditions for f to be starlike of order β in U. Several other results.

Keywords: unit disk, analytic functions, derivative operator.

INTRODUCTION

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}) \quad ,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in C : |z| < 1\}$.

Let be given two functions $f, g \in A, f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathbb{U})$, then their Hadamard product $f(z) * g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U}).$$

For several functions $f_1(z), \ldots, f_m(z) \in A$, we can write in the form

$$f_1(z)*\dots*f_m(z)=z+\sum_{k=2}^{\infty}(a_{1k}\dots a_{mk})z^k,\quad (z\in\mathbb{U}).$$

Let, $(x)_k$ denotes the Pochhammer symbol defined by

$$(x)_k=\begin{cases}1 & for\quad k=0,\\ x(x+1)(x+2)\dots(x+k-1) & for\quad k\in\mathbb{N}=\{1,2,3,\dots\}.\end{cases}$$

The authors in [14] have recently introduced a new generalized derivative operator $I^m(\lambda_1,\lambda_2,l,n)f(z)$ as the following:

Definition 1.1

For $f\in A$ the operator $I^m(\lambda_1,\lambda_2,l,n)$ is defined by $I^m(\lambda_1,\lambda_2,l,n):A\to A$ and let

$$\phi(z):=\frac{1+l-\lambda_1}{1+l}\frac{z}{1-z}+\frac{\lambda_1}{1+l}\frac{z}{(1-z)^2},$$

and

$$F_1(z)=\underbrace{\phi(z)*\dots*\phi(z)}_{(m-1)-times}*\left[\frac{z}{(1-z)^{n+1}}\right].$$

Let also

$$\psi(z):=(1-\lambda_2)\frac{z}{1-z}+\lambda_2\frac{z}{(1-z)^2},$$

and

$$\psi(z)*\psi^{-1}(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}a_k z^k,$$

let

$$F_2(z)=\underbrace{\psi^{-1}(z)*\dots*\psi^{-1}(z)}_{(m)-times}*f(z).$$

Thus we have

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=F_1(z)*F_2(z)$$

$$I^m(\lambda_1,\lambda_2,l,n)f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k)a_k z^k,$$
(2)

where $n, m \in \mathbb{N}_0 = \{0,1,2,...\}$, and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Special cases of this operator includes:

- the Ruscheweyh derivative operator [1] in the cases:

$$I^1(\lambda_1,0,l,n) \equiv I^1(\lambda_1,0,0,n) \equiv I^1(0,0,l,n) \equiv I^0(0,\lambda_2,0,n)$$

$$\equiv I^0(0,0,0,n) \equiv I^{m+1}(0,0,l,n) \equiv I^{m+1}(0,0,0,n) \equiv R^n,$$

- the S$\hat{a}$ l$\hat{a}$ gean derivative operator [2]:

$$I^{m+1}(1,0,0,0) \equiv S^n,$$

- the generalized Ruscheweyh derivative operator [3]:

$$I^2(\lambda_1,0,0,n) \equiv R^n_\lambda,$$

- the generalized S$\hat{a}$ l$\hat{a}$ gean derivative operator introduced by Al-Oboudi [4]:

$$I^{m+1}(\lambda_1,0,0,0) \equiv S^n_\beta,$$

- the generalized Al-Shaqsi and Darus derivative operator[5]:

$$I^{m+1}(\lambda_1,0,0,n) \equiv D^n_{\lambda,\beta},$$

- the Al-Abbadi and Darus generalized derivative operator [6]:

$$I^m(\lambda_1,\lambda_2,0,n) \equiv \mu^{n,m}_{\lambda_1,\lambda_2},$$

and finally

- the Catas derivative operator [7]:

$$I^m(\lambda_1,0,l,n) \equiv I^m(\lambda,\beta,l).$$

Using simple computation one obtains the next result.

$$(l+1)I^{m+1}(\lambda_1,\lambda_2,l,n)f(z) = (1+l-\lambda_1)[I^m(\lambda_1,\lambda_2,l,n) * \varphi^1(\lambda_1,\lambda_2,l)(z)]f(z) +$$

$$\lambda_1 z[(I^m(\lambda_1,\lambda_2,l,n) * \varphi^1(\lambda_1,\lambda_2,l)(z)]',$$

where $(z \in \mathbb{U})$ and $\varphi^1(\lambda_1,\lambda_2,l)(z)$ analytic function given by

$$\varphi^1(\lambda_1,\lambda_2,l)(z) = z + \sum_{k=2}^{\infty} \frac{1}{(1+\lambda_2(k-1))} z^k.$$

Many other work on analytic functions related to derivative operator and integral operator can be read in [15,16,17,18]. There are times, functions are associated with linear operators and create new classes (see for example [9]). Many results are considered with numerous properties are solved and obtained.

Definition 1.2

A function f belonging to A is said to be in the class $S(\alpha)$ in $\mathbb{U}$ if it satisfies

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}),$$

for some $0 \le \alpha < 1$.

Definition 1.3

A function f belonging to A is said to be in the class $C(\alpha)$ in $\mathbb{U}$ if it satisfies

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}),$$

for some $0 \le \alpha < 1$.

We note that $f \in C(\alpha)$ if and only if $zf'(z) \in S(\alpha)$.

Definition 1.4

In [8], for functions $f \in A$ such that $\nu > 0$,

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\nu}\right\} > 0 \quad (z \in \mathbb{U}, \nu > 0),$$

a class of Bazilevic type functions B^{ν} was considered and certain properties were studied.

Definition 1.5

In [9], for functions $f \in A$ such that $\nu > 0$,

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\nu} - \alpha\right\} \ge M\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\nu} - 1\right|, \forall \quad (z \in \mathbb{U}, M \ge 0, 0 \le \alpha < 1),$$

a class M-uniformly Bazilevic type functions $UB_M^{\nu}(\alpha)$ was considered and studied.

Note that $\nu = 0$ gives the subclass M-uniformly starlike $US(\alpha)$

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge M\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad (z \in \mathbb{U}).$$

Now we define a subclass $UB_M^{v,m}(\lambda_1,\lambda_2,l,n,\alpha)$ involving our new generalized derivative operator (2) as follows:

$$\Re\left\{\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^v-\alpha\right\}$$

$$\geq M\left|\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^v-1\right|,\quad (z\in\mathbb{U}).$$

We see that

$$UB_M^{v,1}(\lambda_1,0,l,0,\alpha)\equiv UB_M^v(\alpha),$$

$$UB_M^{0,1}(\lambda_1,0,l,n,\alpha)\equiv R_n(\alpha,M),$$

see [10, 11].

Also, we have

$$UB_M^{0,1}(\lambda_1,0,l,n,0)\equiv R_n(0,M),$$

see [12, 13].

1 Coefficient Bounds

Theorem 2.1 A sufficient condition for a function f of the form (1) to be in the class $UB_M^{v,m}(\lambda_1,\lambda_2,l,n,\alpha)$ is

$$\left[\frac{1+M}{1-\alpha}\right]\left[\left(1+\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)|a_k|\right)^v+1\right]\leq 1,$$

where

$$B_k^m(\lambda_1,\lambda_2,l,n):=\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k),$$

for $n,m\in\mathbb{N}_0=\{0,1,2,...\}$, and $\lambda_2\geq\lambda_1\geq 0, l\geq 0.$

Proof:
It suffices to show that

$$M\left|\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^v-1\right|$$

$$-\Re\left\{\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^{\nu}-\alpha\right\}\le 1-\alpha.$$

Let $z\to 1,$ we get

$$M\left|\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^{\nu}-1\right|$$

$$-\Re\left\{\frac{z[I^m(\lambda_1,\lambda_2,l,n)f(z)]'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^{\nu}-\alpha\right\}$$

$$\le(1+M)\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f(z))'}{I^m(\lambda_1,\lambda_2,l,n)f(z)}\left(\frac{I^m(\lambda_1,\lambda_2,l,n)f(z)}{z}\right)^{\nu}-1\right|$$

$$\le(1+M)\left|(I^m(\lambda_1,\lambda_2,l,n)f(z))'\left(I^m(\lambda_1,\lambda_2,l,n)f(z)\right)^{\nu-1}-1\right|$$

$$\le(1+M)\left|\left(1+\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)a_k\right)\left(1+\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)a_k\right)^{\nu-1}-1\right|$$

$$\le(1+M)\left|\left(1+\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)a_k\right)^{\nu}-1\right|$$

$$\le(1+M)\left|\left(1+\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)|a_k|\right)^{\nu}+1\right|.$$

This last expression is bounded above by $1-\alpha$ if

$$\left[\frac{1+M}{1-\alpha}\right]\left[\left(1+\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)|a_k|\right)^{\nu}+1\right]\le 1,$$

where

$$B_k^m(\lambda_1,\lambda_2,l,n):=\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k).$$

This ends the proof.

Next, we find the coefficient bounds for the class

$$NUB_M^{v,m}(\lambda_1,\lambda_2,l,n,\alpha):=UB_M^{v,m}(\lambda_1,\lambda_2,l,n,\alpha)\cap N,$$

where N is the class of analytic functions takes the form

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k, \quad (z\in\mathbb{U}),$$

$$=z-\sum_{k=2}^{\infty}b_k z^k, \quad (z\in\mathbb{U}).$$

Theorem 2.2 A sufficient condition for a function

$$f(z)=z-\sum_{k=2}^{\infty}b_k z^k, \quad (z\in\mathbb{U}),$$

to be in the class $NUB_M^{v,m}(\lambda_1,\lambda_2,l,n,\alpha)$ is

$$\left[\frac{1+M}{1-\alpha}\right]\left[\left(1-\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)b_k\right)^v+1\right]\le 1,$$

where

$$B_k^m(\lambda_1,\lambda_2,l,n):=\frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m}c(n,k),$$

for $n,m\in\mathbb{N}_0=\{0,1,2,...\}$, and $\lambda_2\ge\lambda_1\ge 0, l\ge 0$.

Proof: We have

$$(1+M)\left|\left(1-\sum_{k=2}^{\infty}kB_k^m(\lambda_1,\lambda_2,l,n)b_k\right)\left(1-\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)b_k\right)^{v-1}-1\right|$$

$$\le(1+M)\left|\left(1-\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)b_k\right)^v-1\right|$$

$$\le(1+M)\left[\left|\left(1-\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)b_k\right)^v\right|+1\right].$$

The last expression is bounded above by $1-\alpha$ if

$$\left[\frac{1+M}{1-\alpha}\right]\left[\left|\left(1-\sum_{k=2}^{\infty}B_k^m(\lambda_1,\lambda_2,l,n)b_k\right)^v\right|+1\right]\le 1,$$

where

$$B_k^m(\lambda_1,\lambda_2,l,n) := \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k).$$

This ends the proof.

CONCLUSION

The main impact of this research work is to motivate to construct new classes *Bazilevic* functions belonging the disk U and study their various properties.

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