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Exact solution for local fractional Diffusion and Wave Equations on Cantor Sets

Ahmad. A. H. Mtawal¹, Eman. A. Maity²

^{1,2}Department of Mathematics, Faculty of Education Almarj, Benghazi University,
Almarj, Libya



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الملخص :

في هذا البحث تم استخدام طريقة الاضطراب التماثلي لموضع الرتب الكسرية وتحويل Sumudu لموضع الرتب الكسرية لدراسة المعادلات الانتشارية والموجة المحددة في مجموعات الكانتور مع شروط ابتدائية ومع المشتقات لموضع الرتب الكسرية. الطريقة التحليلية (LFSHPM) تقلل من الحجم الحسابي ويمكن تطبيقها مباشرة على المعادلات التفاضلية ذات الرتب الكسرية دون اللجوء لأي افتراضات معينة مثل الافتراضات الخطية، تقدير المتغيرات، أو تحويل ما، أو افتراضات مقيدة. توفر المتسلسلة حلول تتقارب بسرعة في عدد قليل من التكرارات. وتم تطبيق الطريقة التحليلية المقترحة بنجاح على المعادلات الانتشارية والموجة المحددة في مجموعات الكانتور ذات الشروط الابتدائية، وأثبتت أنها عالية الكفاءة في الدقة الحسابية. الكلمات المفتاحية: طريقة الاضطراب التماثلي لموضع الرتب الكسرية، تحويل Sumudu لموضع الرتب الكسرية، معادلة الانتشار، معادلة الموجة.

Research Summary:

In this paper, the local fractional homotopy perturbation method and the local fractional Sumudu transform are used to study diffusion and wave equations defined on Cantor sets with the fractal conditions with local fractional derivatives. The LFSHPM analytical method minimizes the computational size and may be applied directly to fractional differential equations without any linearization, discretization of variables, transformation, or restrictive assumptions. It provides series solutions that converge quickly in a few iterations. The proposed analytical method is successfully applied to diffusion and wave equations defined on cantor sets with fractal conditions, and proved to be highly efficient and computational accurate.

Key words: local fractional homotopy perturbation method, local fractional Sumudu transform, Diffusion equation, Wave equation.

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Abstract: To solve the diffusion and wave equations on the Cantor set, the local fractional Sumudu homotopy perturbation method (LFSHPM) is used. In the local sense, the operators are used. The local fractional Sumudu homotopy perturbation method, which is the coupling method of the local fractional homotopy perturbation method and the Sumudu transform, is used to obtain non-differentiable approximate solutions. Illustrative examples are provided to show the new algorithm's excellent accuracy and fast convergence.

1. Introduction

In the last fifteen years, local fractional calculus has played a significant role in fields ranging from fundamental research to engineering [1–3], and it has been used to a diverse set of complicated issues in physics, biology, mechanics, and transdisciplinary domains [4,5]. Various methods, for example, the Adomian decomposition method [6], the variational iteration method [7,8], the homotopy perturbation method [9-12], the fractional iteration method [13, 14], the fractal Laplace and Fourier transforms [15], the Sumudu decomposition method [16], the homotopy perturbation Sumudu transform method [17,18], the homotopy analysis method [19], the heat balance integral method [20], and the fractional variational iteration method [21].

Many processes in science and engineering rely on diffusion equations, such as the diffusion of dissolved substances in solvent liquids, neutrons in a nuclear reactor, and Brownian motion, whereas wave equations describe the motion of a vibrating string (see [22, 23] and the references therein).

The diffusion equation for Cantor sets was recently presented in [24] as "local fractional diffusion equation."

$$\frac{\partial^{\sigma} u(x,t)}{\partial t^{\sigma}} - a^{2\sigma} \frac{\partial^{2\sigma} u(x,t)}{\partial x^{2\sigma}} = 0, \quad (1)$$

where $a^{2\sigma}$ is the fractal diffusion constant, which is a measure of the spreading efficiency of the underlying material, and the local fractional wave equation is stated as follows [25, 26]:

$$\frac{\partial^{2\sigma} u(x,t)}{\partial t^{2\sigma}} - a^{2\sigma} \frac{\partial^{2\sigma} u(x,t)}{\partial x^{2\sigma}} = 0, \quad (2)$$

The following is how the paper is structured. Section 2 introduces the principles of local fractional calculus theory that will be employed in this study. The local fractional Sumudu

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homotopy perturbation method is presented in Section 3. The solutions to the diffusion and wave equations in Cantor set conditions are presented in Section 4. Our conclusions are presented in Section 5.

2. Local Fractional Derivative and Local Fractional Sumudu Transform

Definition 1. [25,26] In fractal space, let $f(x) \in C_\sigma(\alpha, \beta)$, local fractional derivative of $f(x)$ of order σ the point $x = x_0$ is given by

$$D_x^\sigma f(x_0) = \frac{d^\sigma}{dx^\sigma} f(x) \Big|_{x=x_0} = f^{(\sigma)}(x_0) = \lim_{x \rightarrow x_0} \frac{\Delta^\sigma (f(x) - f(x_0))}{(x - x_0)^\sigma}, \quad (3)$$

Where $\Delta^\sigma (f(x) - f(x_0)) \cong \Gamma(\sigma + 1) (f(x) - f(x_0))$.

Definition 2. [25,26] A partition of the interval $[\alpha, \beta]$ is denoted by (t_j, t_{j+1}) , $j = 0, \dots, N - 1$, $t_0 = \alpha$ and $t_N = \beta$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots\}$. Local fractional integral of $f(x)$ in the interval $[\alpha, \beta]$ is given by

$${}_a I_\beta^\sigma f(x) = \frac{1}{\Gamma(\sigma + 1)} \int_\alpha^\beta f(t) (dt)^\sigma = \frac{1}{\Gamma(\sigma + 1)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\sigma. \quad (4)$$

Definition 3. [25, 26] In fractal space, the Mittag-Leffler function, the hyperbolic sine and hyperbolic cosine are defined as

$$E_\sigma(x^\sigma) = \sum_{m=0}^{\infty} \frac{x^{m\sigma}}{\Gamma(m\sigma + 1)}, \quad 0 < \sigma \leq 1, \quad (5)$$

$$\sinh_\sigma(x^\sigma) = \sum_{m=0}^{\infty} \frac{x^{(2m+1)\sigma}}{\Gamma((2m+1)\sigma + 1)}, \quad 0 < \sigma \leq 1, \quad (6)$$

$$\cosh_\sigma(x^\sigma) = \sum_{m=0}^{\infty} \frac{x^{2m\sigma}}{\Gamma(2m\sigma + 1)}, \quad 0 < \sigma \leq 1. \quad (7)$$

By using the local fractional derivative (3) and the equation (5) it can be easily shown that

$$\frac{d^\sigma}{dx^\sigma} \left(\frac{x^{m\sigma}}{\Gamma(m\sigma + 1)} \right) = \frac{x^{(m-1)\sigma}}{\Gamma((m-1)\sigma + 1)}. \quad (8)$$

Definition 4. [27, 28] The local fractional Sumudu transform of $f(x)$ is defined by

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$${}^{lf} S_{\sigma} [f(x)] = G_{\sigma}(\delta) = \frac{1}{\Gamma(\sigma+1)} \int_0^{\infty} E_{\sigma}(-\delta^{\sigma} x^{\sigma}) \frac{f(x)}{\delta^{\sigma}} (dx)^{\sigma}, 0 < \sigma \leq 1. \quad (9)$$

The following inverse formula of (9) is defined as

$${}^{lf} S_{\sigma}^{-1} [G_{\sigma}(\delta)] = f(x), 0 < \sigma \leq 1. \quad (10)$$

Properties of local fractional Sumudu Transform

$${}^{lf} S_{\sigma} \left[\frac{x^{\sigma}}{\Gamma(\sigma+1)} \right] = \delta^{\sigma}, \quad (11)$$

Theorem 1.

1. Local fractional Sumudu transform of local fractional derivative is defined by

$$\begin{aligned} {}^{lf} S_{\sigma} \left[\frac{d^m \sigma f(x)}{dx^{m\sigma}} \right] &= \frac{1}{\delta^{m\sigma}} \left[G_{\sigma}(\delta) - \sum_{k=0}^{m-1} \delta^{k\sigma} f^{(k\sigma)}(0) \right] \\ &= \frac{G_{\sigma}(\delta)}{\delta^{m\sigma}} - \frac{1}{\delta^{m\sigma}} \sum_{k=0}^{m-1} \delta^{k\sigma} \frac{\partial^{k\sigma} f(0)}{\partial t^{k\sigma}}. \end{aligned} \quad (12)$$

2. Local fractional Sumudu transform of local fractional integral is defined by

$${}^{lf} S_{\sigma} \left[{}_0 I_x^{\sigma} f(x) \right] = \delta^{\sigma} G_{\sigma}(\delta). \quad (13)$$

3. Local Fractional Sumudu Homotopy perturbation Method

Consider the local fractional partial differential equations shown below:

$$L_{\sigma} u(x, t) + R_{\sigma} u(x, t) = g_{\sigma}(x, t), \quad (14)$$

Where L_{σ} is the linear local fractional operator, R_{σ} is the linear local fractional operator of order the last then L_{σ} and $g_{\sigma}(x, t)$ is given function.

By applying a local fractional Sumudu transform (denoted in this paper by ${}^{lf} S_{\sigma}$) on each side of equation (14), we get

$${}^{lf} S_{\sigma} [L_{\sigma} u(x, t)] = {}^{lf} S_{\sigma} [g(x, t)] - {}^{lf} S_{\sigma} [R_{\sigma} u(x, t)]. \quad (15)$$

We have based on the properties of this transform.

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$${}^{\text{lf}} S_{\sigma} [u(x, t)] = {}^{\text{lf}} S_{\sigma} [g(x, t) - R_{\sigma} u(x, t)].$$

$$G_{\sigma}(\delta) = \sum_{k=0}^{m-1} \delta^{k\sigma} \frac{\partial^{k\sigma} u(x, 0)}{\partial t^{k\sigma}} + \delta^{m\sigma} {}^{\text{lf}} S_{\sigma} [g(x, t) - R_{\sigma} u(x, t)]. \quad (16)$$

Operating with the Sumudu inverse on both sides of Equation (16) gives

$$u(x, t) = \sum_{k=0}^{m-1} \frac{\partial^{k\sigma} u(x, 0)}{\partial t^{k\sigma}} \frac{t^{k\sigma}}{\Gamma(k\sigma + 1)} + {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{m\sigma} {}^{\text{lf}} S_{\sigma} [g(x, t) - R_{\sigma} u(x, t)] \right]. \quad (17)$$

Now we apply the homotopy perturbation method (HPM).

$$u(x, t) = \sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t). \quad (18)$$

Substituting Equation (17) in Equation (18), we get

$$\sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t) = \sum_{k=0}^{m-1} \frac{\partial^{k\sigma} u(x, 0)}{\partial t^{k\sigma}} \frac{t^{k\sigma}}{\Gamma(k\sigma + 1)} + P^{n\sigma} {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{m\sigma} {}^{\text{lf}} S_{\sigma} \left[g(x, t) - R_{\sigma} \left(\sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t) \right) \right] \right]. \quad (19)$$

Equating the terms with identical powers of P^{σ} , we can obtain a series of equations as the follows:

$$P^0 : u_0(x, t) = \sum_{k=0}^{m-1} \frac{\partial^{k\sigma} u(x, 0)}{\partial t^{k\sigma}} \frac{t^{k\sigma}}{\Gamma(k\sigma + 1)} + {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{m\sigma} {}^{\text{lf}} S_{\sigma} [g(x, t)] \right].$$

$$P^{\sigma} : u_1(x, t) = -{}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{m\sigma} {}^{\text{lf}} S_{\sigma} [R_{\sigma} u_0(x, t)] \right],$$

$$\vdots$$

$$P^{n\sigma} : u_n(x, t) = -{}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{m\sigma} {}^{\text{lf}} S_{\sigma} [R_{\sigma} u_{n-1}(x, t)] \right], \quad (20)$$

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proceeding in the same manner, the rest of the components $u_n(x, t)$ can be completely found and the series solution is thus entirely determined. We approximate the analytical solution $u(x, t)$ by truncated series as:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t). \quad (21)$$

4. Applications

In this section, four examples for diffusion and wave equations on cantor sets will demonstrate the efficiency of local fractional Sumudu homotopy perturbation method

Example 1. Consider the following diffusion equation on Cantor sets:

$$\frac{\partial^\sigma u(x, t)}{\partial t^\sigma} - \frac{\partial^{2\sigma} u(x, t)}{\partial x^{2\sigma}} = 0, \quad 0 < \sigma \leq 1, \quad (22)$$

subject to the initial condition

$$u(x, 0) = \frac{x^\sigma}{\Gamma(\sigma+1)}. \quad (23)$$

We structure the iterative relation using relation (19) as

$$\sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t) = u(x, 0) + P^{n\sigma} \mathcal{L} S_\sigma^{-1} \left[\delta^\sigma \mathcal{L} S_\sigma \left[\left(\sum_{n=0}^{\infty} P^{n\sigma} \frac{\partial^{2\sigma} u_n(x, t)}{\partial x^{2\sigma}} \right) \right] \right]. \quad (24)$$

The following approximations are generated by comparing the coefficients with similar powers of P^σ in Eq. (24):

$$\begin{aligned} P^{0\sigma} : u_0(x, t) &= \frac{x^\sigma}{\Gamma(\sigma+1)}, \\ P^\sigma : u_1(x, t) &= \mathcal{L} S_\sigma^{-1} \left[\delta^\sigma \mathcal{L} S_\sigma \left[\frac{\partial^{2\sigma} u_0(x, t)}{\partial x^{2\sigma}} \right] \right] = 0, \\ P^{2\sigma} : u_2(x, t) &= \mathcal{L} S_\sigma^{-1} \left[\delta^\sigma \mathcal{L} S_\sigma \left[\frac{\partial^{2\sigma} u_1(x, t)}{\partial x^{2\sigma}} \right] \right] = 0, \\ P^{3\sigma} : u_3(x, t) &= \mathcal{L} S_\sigma^{-1} \left[\delta^\sigma \mathcal{L} S_\sigma \left[\frac{\partial^{2\sigma} u_2(x, t)}{\partial x^{2\sigma}} \right] \right] = 0, \dots \end{aligned} \quad (25)$$

and so on.

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The local fractional series solution is hence

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t) = \frac{x^{\sigma}}{\Gamma(\sigma+1)}. \quad (26)$$

The result is the same as the one which is obtained by the local fractional series expansion method and local fractional Laplace variational iteration method [29, 30].

Example 2. Consider the following diffusion equation on Cantor sets:

$$\frac{\partial^{\sigma} u(x, t)}{\partial t^{\sigma}} - \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u(x, t)}{\partial x^{2\sigma}} = 0, \quad 0 < \sigma \leq 1, \quad (27)$$

subject to the initial condition

$$u(x, 0) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)}. \quad (28)$$

We structure the iterative relation using relation (19) as

$$\sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t) = u(x, 0) + P^{n\sigma} I_{\sigma}^{-1} \left[\delta^{\sigma} I_{\sigma} \left[\left(\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{n=0}^{\infty} P^{n\sigma} \frac{\partial^{2\sigma} u_n(x, t)}{\partial x^{2\sigma}} \right) \right] \right]. \quad (29)$$

The following approximations are generated by comparing the coefficients with similar powers of P^{σ} in Eq. (29):

$$P^{0\sigma} : u_0(x, t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)},$$

$$\begin{aligned} P^{\sigma} : u_1(x, t) &= I_{\sigma}^{-1} \left[\delta^{\sigma} I_{\sigma} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_0(x, t)}{\partial x^{2\sigma}} \right] \right] \\ &= I_{\sigma}^{-1} \left[\delta^{\sigma} I_{\sigma} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \right] \right] \\ &= I_{\sigma}^{-1} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \delta^{\sigma} \right] = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{\sigma}}{\Gamma(\sigma+1)}, \end{aligned}$$

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$$\begin{aligned}
 P^{2\sigma} : u_2(x, t) &= {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{\sigma} {}^{\text{lf}} S_{\sigma} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_1(x, t)}{\partial x^{2\sigma}} \right] \right] \\
 &= {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{\sigma} {}^{\text{lf}} S_{\sigma} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{\sigma}}{\Gamma(\sigma+1)} \right] \right] \\
 &= {}^{\text{lf}} S_{\sigma}^{-1} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \delta^{2\sigma} \right] \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)}, \\
 P^{3\sigma} : u_3(x, t) &= {}^{\text{lf}} S_{\sigma}^{-1} \left[\delta^{\sigma} {}^{\text{lf}} S_{\sigma} \left[\frac{\partial^{2\sigma} u_2(x, t)}{\partial x^{2\sigma}} \right] \right]; \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)},
 \end{aligned} \tag{30}$$

and so on.

The local fractional series solution is hence

$$\begin{aligned}
 u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \left(1 + \frac{t^{\sigma}}{\Gamma(\sigma+1)} + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \dots \right) \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{k=0}^{\infty} \frac{t^{k\sigma}}{\Gamma(k\sigma+1)} \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} E_{\sigma}(t^{\sigma}).
 \end{aligned} \tag{31}$$

The result is the same as the one which is obtained local fractional series expansion method and local fractional Laplace variational iteration method [29, 30].

Example 3. Consider the following wave equation on Cantor sets:

$$\frac{\partial^{2\sigma} u(x, t)}{\partial t^{2\sigma}} - \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u(x, t)}{\partial x^{2\sigma}} = 0, \quad 0 < \sigma \leq 1, \tag{32}$$

subject to the initial condition

$$u(x, 0) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)}, \quad \frac{\partial^{\sigma} u(x, 0)}{\partial t^{\sigma}} = 0. \tag{33}$$

We structure the iterative relation using relation (19) as

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$$\sum_{n=0}^{\infty} P^{n\sigma} u_n(x,t) = u(x,0) + \frac{\partial^\sigma u(x,0)}{\partial t^\sigma} \frac{t^\sigma}{\Gamma(\sigma+1)} + P^{n\sigma} \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\left(\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{n=0}^{\infty} P^{n\sigma} \frac{\partial^{2\sigma} u_n(x,t)}{\partial x^{2\sigma}} \right) \right] \right]. \quad (34)$$

The following approximations are generated by comparing the coefficients with similar powers of P^σ in Eq. (34):

$$P^{0\sigma} : u_0(x,t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)},$$

$$\begin{aligned} P^\sigma : u_1(x,t) &= \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_0(x,t)}{\partial x^{2\sigma}} \right] \right] \\ &= \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \right] \right] \\ &= \text{I} S_\sigma^{-1} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \delta^{2\sigma} \right] \\ &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)}, \end{aligned}$$

$$\begin{aligned} P^{2\sigma} : u_2(x,t) &= \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_1(x,t)}{\partial x^{2\sigma}} \right] \right] \\ &= \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} \right] \right] \\ &= \text{I} S_\sigma^{-1} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \delta^{4\sigma} \right] \\ &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{4\sigma}}{\Gamma(4\sigma+1)}, \end{aligned}$$

$$\begin{aligned} P^{3\sigma} : u_3(x,t) &= \text{I} S_\sigma^{-1} \left[\delta^{2\sigma} \text{I} S_\sigma \left[\frac{\partial^{2\sigma} u_2(x,t)}{\partial x^{2\sigma}} \right] \right] \vdots \\ &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{6\sigma}}{\Gamma(6\sigma+1)}, \end{aligned} \quad (35)$$

and so on.

The local fractional series solution is hence

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \left(1 + \frac{t^{2\sigma}}{\Gamma(2\sigma+1)} + \frac{t^{4\sigma}}{\Gamma(4\sigma+1)} + \frac{t^{6\sigma}}{\Gamma(6\sigma+1)} + \dots \right) \\ &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{k=0}^{\infty} \frac{t^{2k\sigma}}{\Gamma(2k\sigma+1)} \\ &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \cosh_{\sigma}(t^{\sigma}). \end{aligned} \quad (36)$$

The result is the same as the one which is obtained by local fractional series expansion method and local fractional Laplace variational iteration method [29, 30].

Example 4. Consider the following wave equation on Cantor sets:

$$\frac{\partial^{2\sigma} u(x, t)}{\partial t^{2\sigma}} - \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u(x, t)}{\partial x^{2\sigma}} = 0, \quad 0 < \sigma \leq 1, \quad (37)$$

subject to the initial condition

$$u(x, 0) = 0, \quad \frac{\partial^{\sigma} u(x, 0)}{\partial t^{\sigma}} = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)}. \quad (38)$$

We structure the iterative relation using relation (19) as

$$\begin{aligned} \sum_{n=0}^{\infty} P^{n\sigma} u_n(x, t) &= u(x, 0) + \frac{\partial^{\sigma} u(x, 0)}{\partial t^{\sigma}} \frac{t^{\sigma}}{\Gamma(\sigma+1)} \\ &\quad + P^{n\sigma} I_{\sigma}^{-1} \left[\delta^{2\sigma} I_{\sigma} \left[\left(\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{n=0}^{\infty} P^{n\sigma} \frac{\partial^{2\sigma} u_n(x, t)}{\partial x^{2\sigma}} \right) \right] \right]. \end{aligned} \quad (39)$$

The following approximations are generated by comparing the coefficients with similar powers of P^{σ} in Eq. (39):

$$P^{0\sigma} : u_0(x, t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{\sigma}}{\Gamma(\sigma+1)},$$

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$$\begin{aligned}
 P^\sigma : u_1(x, t) &= {}^{\text{lf}} S_\sigma^{-1} \left[\delta^{2\sigma} {}^{\text{lf}} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_0(x, t)}{\partial x^{2\sigma}} \right] \right] \\
 &= {}^{\text{lf}} S_\sigma^{-1} \left[\delta^{2\sigma} {}^{\text{lf}} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^\sigma}{\Gamma(\sigma+1)} \right] \right] \\
 &= {}^{\text{lf}} S_\sigma^{-1} \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \delta^{3\sigma} \right] \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{3\sigma}}{\Gamma(3\sigma+1)}, \\
 P^{2\sigma} : u_2(x, t) &= {}^{\text{lf}} S_\sigma^{-1} \left[\delta^{2\sigma} {}^{\text{lf}} S_\sigma \left[\frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{\partial^{2\sigma} u_1(x, t)}{\partial x^{2\sigma}} \right] \right] \\
 P^{3\sigma} : u_3(x, t) &= {}^{\text{lf}} S_\sigma^{-1} \left[\delta^{2\sigma} {}^{\text{lf}} S_\sigma \left[\frac{\partial^{2\sigma} u_2(x, t)}{\partial x^{2\sigma}} \right] \right] \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \frac{t^{7\sigma}}{\Gamma(7\sigma+1)}, \dots
 \end{aligned} \tag{40}$$

and so on.

The local fractional series solution is hence

$$\begin{aligned}
 u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) = \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \left(\frac{t^\sigma}{\Gamma(\sigma+1)} + \frac{t^{3\sigma}}{\Gamma(3\sigma+1)} + \frac{t^{5\sigma}}{\Gamma(5\sigma+1)} + \frac{t^{7\sigma}}{\Gamma(7\sigma+1)} + \dots \right) \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sum_{k=0}^{\infty} \frac{t^{(2k+1)\sigma}}{\Gamma((2k+1)\sigma+1)} \\
 &= \frac{x^{2\sigma}}{\Gamma(2\sigma+1)} \sinh_\sigma(t^\sigma).
 \end{aligned} \tag{41}$$

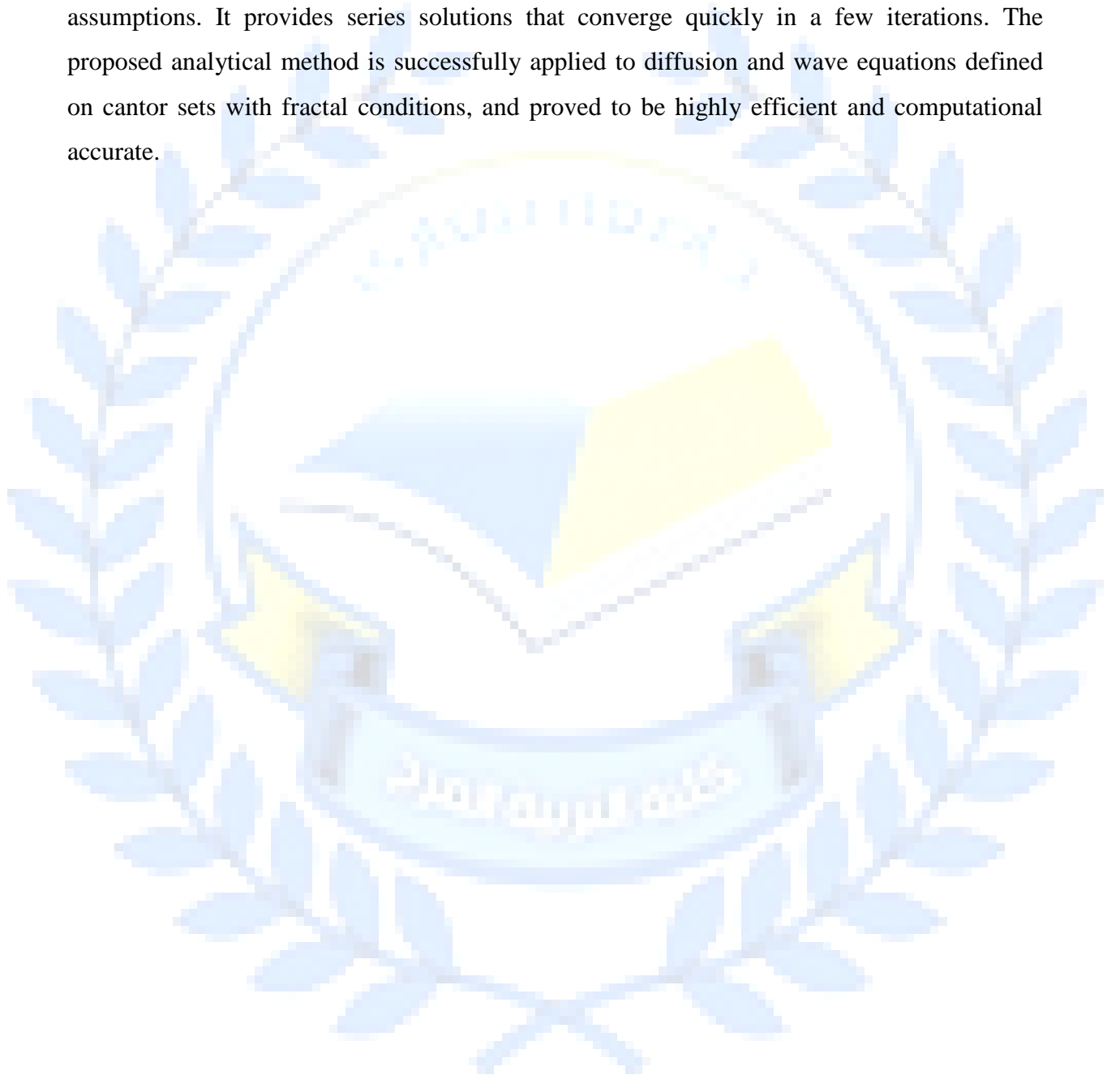
The result is the same as the one which is obtained by the Local fractional variational iteration and Decomposition methods and local fractional Laplace variational iteration method [30, 31].

5. Conclusions

In this paper, the local fractional homotopy perturbation method and the local fractional Sumudu transform are used to study diffusion and wave equations defined on Cantor sets

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with the fractal conditions with local fractional derivatives. The LFSHPM analytical method minimizes the computational size and may be applied directly to fractional differential equations without any linearization, discretization of variables, transformation, or restrictive assumptions. It provides series solutions that converge quickly in a few iterations. The proposed analytical method is successfully applied to diffusion and wave equations defined on cantor sets with fractal conditions, and proved to be highly efficient and computational accurate.



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