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On the existence of a unique solution of a coupled system of differential equations with the nonlocal two-point boundary conditions

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Abstract:

The aim of the paper is to study the existence of a unique solution of a nonlocal problem of coupled system of differential equations with nonlocal two-point boundary conditions. By means of Banach fixed point theorem, an existence result of a solution is proved for the nonlocal problem considered.

Keywords: Coupled system, nonlocal conditions, a unique solution, Banach fixed point theorem.

الملخص:

الهدف من البحث هو دراسة وجود حل وحيد لمشكلة غير محلية مكونة من نظام مزدوج من المعادلات التفاضلية مع شروط حدية ثنائية النقط غير محلية. باستخدام نظرية النقطة الثابتة لبناخ تم أثبات وجود حل للمشكلة غير المحلية قيد الدراسة.

الكلمات المفتاحية: نظام مزدوج، شروط غير محلية، حل وحيد، نظرية النقطة الثابتة لبناخ.

I. Introduction

A number of studies on the existence of solutions for several classes of problems with nonlocal conditions have been carried out ([1-4], [6-13] and [15]) .

In [8] the authors studied the uniqueness solutions of the nonlocal problem

$$\frac{dx}{dt} = f_1(t, y(t)), \qquad t \in (0, T]$$

$$\frac{dy}{dt} = f_2(t, x(t)), \qquad t \in (0, T]$$

with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0,\ a_k > 0, \tau_k \in (0, T)$$

$$y(0) + \sum_{j=1}^{m} b_j x(\eta_j) = y_0,\ b_j > 0, \eta_j \in (0, T).$$

Also, in [15] the authors studied the existence of at least one solution of the nonlocal problem

$$\frac{dx}{dt} = f_1(t, y(\varphi_1(t))), \qquad t \in (0, 1] \tag{1}$$

$$\frac{dy}{dt} = f_2(t, x(\varphi_2(t))), \qquad t \in (0, 1] \tag{2}$$

with the nonlocal two-point boundary conditions

$$x(\tau) + \alpha x(\eta) = 0\,, \tag{3}$$

$$y(\tau) + \beta y(\eta) = 0, \tag{4}$$

where $\tau \neq \eta$, $\tau, \eta \in [0, 1]$ and $\alpha, \beta > 0$.

In the present paper we study the existence of a unique solution of the nonlocal problem (1) – (4).

II. Notation and basic facts

In this section we collect definitions and auxiliary facts which will be needed further on**.**

(1) Let $C = C(I)$ denotes the class of continuous functions defined on the interval $I = [a, b]$ with the standard norm

$$\|f\| = sup_{t\in[a,b]} |f(t)|.$$

We will accept the following definition of the concept of a contraction operator.

Definition 1. [5] Let T be an operator defined on a set B in a Banach space E, satisfying the condition:

$$\|Tx - Ty\| \leq q\|x - y\|,\ \ x, y \in B.$$

If $0 \leq q \leq 1$, then T is called a contraction operator.

Also, we will only need the following fixed point theorem.

Theorem 1. [14] (Banach fixed point theorem)

Let $T: X \to X$ be a contraction operator, where X is a Banach space and there is a nonnegative real number $q < 1$ such that $\|Tx - Ty\| \leq q\|x - y\|, \quad \forall\, x, y \in X.$ Then the map T admits one and only one fixed point $z \in X$ such that $Tz = z$.

III. Integral Representation

Let X be the class of all ordered pairs (x, y), $x, y \in C(0,1]$ with the norm

$$\|(x,y)\|_X = \|x\| + \|y\| = sup_{t\in[0,1]}|x(t)| + sup_{t\in[0,1]}|y(t)|,$$

and let $\varphi_i : [0,1] \to [0,1]$ be a continuous function for $i = 1,2$.

The nonlocal problem(1) - (4) will be considered under the following assumptions:

(h_1) $f_i : [0,1] \times R \to R$ is continuous for $i = 1,2$.

(h_2) f_i satisfy the Lipschitz condition

$$|f_i(t,x) - f_i(t,y)| \le l_i|x - y|, \qquad l_i > 0, \quad i = 1,2.$$

Then we have the following lemma:

Lemma 3.1 [15] The solution of the nonlocal problem (1) - (4) can be expressed by system of the integral equations

$$x(t) = \int_0^t f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds - \\ -\frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds, \qquad (5)$$

and

$$y(t) = \int_0^t f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds - \\ -\frac{1}{1+\beta}\int_0^\tau f_2\left(s, x(\varphi_2(s))\right) ds. \qquad (6)$$

Proof. Integrating equation (1), we obtain

$$x(t) = x(0) + \int_0^t f_1\left(s, y(\varphi_1(s))\right) ds,$$

$$x(\eta) = x(0) + \int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds$$

and

$$\alpha x(\eta) = \alpha x(0) + \alpha\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds$$

(7)

Let $t = \tau$, then

$$x(\tau) = x(0) + \int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds. \qquad (8)$$

Substituting from equations (7) and (8), we get

$$x(0) = -\frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds .$$

Hence

$$x(t) = \int_0^t f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds - \\ -\frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds.$$

Similarly, Integrating equation (2), we obtain

$$y(t) = y(0) + \int_0^t f_2\left(s, x(\varphi_2(s))\right) ds,$$

$$y(\eta) = y(0) + \int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds,$$

and

$$\beta x(\eta) = \beta x(0) + \beta \int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds. \qquad (9)$$

Let $t = \tau$, then

$$y(\tau) = y(0) + \int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds. \qquad (10)$$

Substituting from equations (9) and (10), we get

$$y(0) = -\frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds .$$

Hence

$$y(t) = \int_0^t f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds.$$

Now, differentiating (5) and (6), we obtain

$$\frac{dx}{dt} = \frac{d}{dt}\int_0^t f_1\left(s, y(\varphi_1(s))\right) ds + 0.$$

This mean that

$$\frac{dx}{dt} = f_1\left(t, y(\varphi_1(t))\right).$$

Similarly,

$$\frac{dy}{dt} = \frac{d}{dt}\int_0^t f_2\left(s, x(\varphi_2(s))\right) ds + 0.$$

This mean that

$$\frac{dy}{dt} = f_2\left(t, x(\varphi_2(t))\right).$$

Also, from (5) and (6), we get

$$x(\tau) = \int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds$$
$$= \frac{\alpha}{1+\alpha}\left(\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds - \int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds\right),$$

and

$$x(\eta) = \int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds$$
$$= \frac{\alpha}{1+\alpha}\left(\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds\right).$$

Hence

$$x(\tau) + \alpha x(\eta) = 0.$$

Similarly

$$y(\tau) = \int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds.$$
$$= \frac{\beta}{1+\beta}\left(\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds\right)$$

and

$$y(\eta) = \int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds.$$
$$= \frac{\beta}{1+\beta}\left(\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds\right).$$

Hence

$$y(\tau) + \beta y(\eta) = 0.$$

This proves the equivalence between the problem (1) – (4) and the integral equations (5) and (6).

Now we are prepared to formulate our main existence result.

IV. Existence theorem

In this section we will study the existence of a unique solution of nonlocal problem (1) - (4).

Theorem 4. 1 Consider that assumptions (h_1) and (h_2) are satisfied if $2l < 1$, where $l = \max(l_1, l_2)$, then the nonlocal problem (1) – (4) has a unique solution $z \in X$.

Proof. Define the operator $\mathcal{F}$ by

$$\mathcal{F}(x,y) = (\mathcal{F}_1 y,\ \mathcal{F}_2 x),$$

where

$$\mathcal{F}_1 y = \int_0^t f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds,$$

$$\mathcal{F}_2 x = \int_0^t f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^\tau f_2\left(s, x(\varphi_2(s))\right) ds.$$

Let $x, y \in C(0,1]$, $t_1, t_2 \in (0,1]$. Then from assumption (h_1) and for every $\epsilon > 0, \delta > 0$ such that $|t_2 - t_1| < \delta$, we have

$$|\mathcal{F}_1 y(t_2) - \mathcal{F}_1 y(t_1)| = \Big|\int_0^{t_2} f_1\left(s, y(\varphi_1(s))\right) ds -$$
$$- \frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds - \int_0^{t_1} f_1\left(s, y(\varphi_1(s))\right) ds +$$
$$+ \frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s, y(\varphi_1(s))\right) ds + \frac{1}{1+\alpha}\int_0^\tau f_1\left(s, y(\varphi_1(s))\right) ds\Big| \le \int_{t_1}^{t_2}\left|f_1\left(s, y(\varphi_1(s))\right)\right| ds.$$

This implies that $\mathcal{F}_1 y \in C(0,1]$.

Similarly, we have

$$|\mathcal{F}_2 x(t_2) - \mathcal{F}_2 x(t_1)| = \Big|\int_0^{t_2} f_2\left(s, x(\varphi_2(s))\right) ds -$$
$$- \frac{\beta}{1+\beta}\int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^\tau f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{t_1} f_2\left(s, x(\varphi_2(s))\right) ds +$$
$$+ \frac{\beta}{1+\beta}\int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds + \frac{1}{1+\beta}\int_0^\tau f_2\left(s, x(\varphi_2(s))\right) ds\Big| \le \int_{t_1}^{t_2}\left|f_2\left(s, x(\varphi_2(s))\right)\right| ds.$$

This implies that $\mathcal{F}_2 x \in C(0,1]$.

Now

$$\mathcal{F}(x(t_2), y(t_1)) - \mathcal{F}(x(t_1), y(t_1)) = (\mathcal{F}_1 y(t_2), \mathcal{F}_2 x(t_2)) - (\mathcal{F}_1 y(t_1), \mathcal{F}_2 x(t_1))$$
$$= (\mathcal{F}_1 y(t_2) - \mathcal{F}_1 y(t_1),\ \mathcal{F}_2 x(t_2) - \mathcal{F}_2 x(t_2)).$$

Therefore,

$\|(\mathcal{F}_1 y(t_2) - \mathcal{F}_1 y(t_1),\ \mathcal{F}_2 x(t_2) - \mathcal{F}_2 x(t_2))\| = \|\mathcal{F}_1 y(t_2) - \mathcal{F}_1 y(t_1)\| + \|\mathcal{F}_2 x(t_2) - \mathcal{F}_2 x(t_2)\|$

Then $\mathcal{F}_1, \mathcal{F}_2 : C(0,1] \to C(0,1]$. Hence $\mathcal{F}: X \to X$.

Let $z = (x,y) \in X$, and let $(u,v) \in X$, then

$$\mathcal{F}(x,y) = (\mathcal{F}_1 y(t),\ \mathcal{F}_2 x(t)), \qquad \mathcal{F}(u,v) = (\mathcal{F}_1 v(t),\ \mathcal{F}_2 u(t)),$$

and

$$|\mathcal{F}_1 y - \mathcal{F}_1 v| = \left|\int_0^t \left(f_1\left(s, y(\varphi_1(s))\right) - f_1\left(s, v(\varphi_1(s))\right)\right) ds - \frac{\alpha}{1+\alpha}\int_0^\eta \left(f_1\left(s, y(\varphi_1(s))\right) - \right.\right.$$
$$\left.\left. - f_1\left(s, v(\varphi_1(s))\right)\right) ds - \frac{1}{1+\alpha}\int_0^\tau \left(f_1\left(s, y(\varphi_1(s))\right) - f_1\left(s, v(\varphi_1(s))\right)\right) ds\right| \leq$$

$$\leq \int_0^t \left|\left(f_1\left(s, y(\varphi_1(s))\right) - f_1\left(s, v(\varphi_1(s))\right)\right)\right| ds + \frac{\alpha}{1+\alpha}\int_0^\eta \left|\left(f_1\left(s, y(\varphi_1(s))\right) - \right.\right.$$
$$\left.\left. - f_1\left(s, v(\varphi_1(s))\right)\right)\right| ds + \frac{1}{1+\alpha}\int_0^\tau \left|\left(f_1\left(s, y(\varphi_1(s))\right) - f_1\left(s, v(\varphi_1(s))\right)\right)\right| ds \leq$$

$$\leq\ l_1\int_0^t \|y - v\|\, ds + \frac{\alpha}{1+\alpha}\ l_1\int_0^\eta \|y - v\| ds + \frac{1}{1+\alpha}\ l_1\int_0^\tau \|y - v\| ds \leq$$
$$\leq\ l_1\|y - v\| + \frac{\alpha}{1+\alpha}\ l_1\|y - v\| + \frac{1}{1+\alpha}\ l_1\|y - v\| \leq 2\ l_1\|y - v\|.$$

Similarly,

$$|\mathcal{F}_2 x - \mathcal{F}_2 u| = \left|\int_0^t \left(f_2\left(s, x(\varphi_2(s))\right) - f_2\left(s, u(\varphi_2(s))\right)\right) ds - \frac{\alpha}{1+\alpha}\int_0^\eta \left(f_2\left(s, x(\varphi_2(s))\right) - \right.\right.$$
$$\left.\left. - f_2\left(s, u(\varphi_2(s))\right)\right) ds - \frac{1}{1+\alpha}\int_0^\tau \left(f_2\left(s, x(\varphi_2(s))\right) - f_2\left(s, u(\varphi_2(s))\right)\right) ds\right| \leq$$

$$\leq \int_0^t \left|\left(f_2\left(s, x(\varphi_2(s))\right) - f_2\left(s, u(\varphi_2(s))\right)\right)\right| ds + \frac{\alpha}{1+\alpha}\int_0^\eta \left|\left(f_2\left(s, x(\varphi_2(s))\right) - \right.\right.$$
$$\left.\left. - f_2\left(s, u(\varphi_2(s))\right)\right)\right| ds + \frac{1}{1+\alpha}\int_0^\tau \left|\left(f_2\left(s, x(\varphi_2(s))\right) - f_2\left(s, u(\varphi_2(s))\right)\right)\right| ds \leq$$

$$\leq\ l_2\int_0^t \|x - u\|\, ds + \frac{\alpha}{1+\alpha}\ l_2\int_0^\eta \|x - u\| ds + \frac{1}{1+\alpha}\ l_2\int_0^\tau \|x - u\| ds \leq$$
$$\leq\ l_2\|x - u\| + \frac{\alpha}{1+\alpha}\ l_2\|x - u\| + \frac{1}{1+\alpha}\ l_2\|x - u\| \leq 2\ l_2\|x - u\|.$$

Hence

$$\|\mathcal{F}(x,y) - \mathcal{F}(u,v)\|_X \leq\ 2l_1\|y - v\| + 2l_2\|x - u\| \leq 2l(\|y - v\| + \|x - u\|) =$$
$$= 2l\ \|(x,y) - (u,v)\|_X.$$

Since $2l < 1$, then $\mathcal{F}$ is a contraction operator.

Using the Banach fixed point theorem we deduce that there exists a unique solution $z \in X$ of the coupled system of the integral equations (5) and (6).

From the equivalence of the problem (1) - (4), and the coupled system of the integral equations (5) and (6) there exists a unique solution of the problem (1) - (4).

This completes the proof. ■

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