University of Benghazi
Faculty of Education Almarj

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والأربعون / أبريل / 2020

On Hyper Ideals of Γ – hypernear-ring

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الملخص:

في العام 2010 قدمت جاما–قريب الحلقة الفائقة بواسطة B. Davvaz [8].

في هذه الورقة العلمية، تم برهنة أن الإتحاد والتقاطع الاختياري لعائلة من المثاليات الفائقة للجاما–قريب الحلقة الفائقة هو مثالي فائق لنفس الجاما–قريب الحلقة الفائقة.

كذلك تم إثبات أن جاما–قريب الحلقة الفائقة المقسمة هي أيضاً تمثل جاما–قريب الحلقة الفائقة.

Abstract:

In 2010, hyperideals of Γ- hypernear-ring introduced by B. Davvaz [8]. In this paper, arbitrary union and intersection of a family of hyperideals of Γ- hypernear-ring is hyperideal of Γ- hypernear-ring are proved. This study showed that, the quotient Γ-hypernear-ring (M/I, ⊕, ⊙) is Γ-hypernear-ring.

Key Words: *near-ring, subnear-ring, hyperoperation, hypergroupoid, hypernear-rings, Γ – hypernear-ring, hyperideal of Γ – hypernear-ring, and Γ – hypernear-ring homomorphism.*

1. Introduction

Algebraic hyperstructures are a generalization of classical algebraic structures, In a classical algebraic structure, the composition of two elements of a set is again an element of the same set, while in an algebraic hyperstructure, the composition of two elements is a non-empty subset of the same set. The theory of hyperstructures has been initiated in 1934 by a French mathematician, Marty introduced hypergroups as a generalization of groups. The hyperstructure theory and its applications have been investigated by the contribution of many mathematician [1] for example, semi-hypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity.
As it is well known, the concept of hypernear-rings was first introduced by Dasic. Bijan Davvaz, Jianming Zhan and Kyung Ho kim, introduced Γ -hypernear-ring which is the generalization of hypernear-rings [2] .

2. Preliminaries

Definition 2.1.[3] A near-ring is a set N together with two binary operations " + " and " . " such that

i) (N , +) is a group (not necessarily abelian)
ii) (N , .) is a semi-group .
iii) for all $a, b, c \in N$: $(a+b).c = c.a + c.b$ (right distributive law)

This near-ring will be termed as right near-ring. If $c.(a+b) = c.a + c.b$ instead of condition iii) the set satisfies, then we call a left near-ring. If (N , +) is abelian, we call an abelian near-ring. If (N , .) is commutative we call itself a commutative near-ring. Clearly if it is commutative near-ring then left and right distributive law is satisfied and is called a commutative near-ring. In this paper the word (be a near-ring) shall mean a right near-ring.

Examples 2.2.

1) The set of integers, the set of all rational numbers, the set of real numbers, and the set of complex numbers are rings as well as near rings with respect to usual addition and multiplication.
2) Consider $Z_8 = \{0,1,2,3,4,5,6,7\}$ with respect to addition and multiplication modulo 8.This is a ring as well as a near-ring.

Remark 2.3. Every ring is a near-ring, but the converse may not be true. We can show that in the next example.

Example 2.4.[4] The triple $(Z_2, +, .)$ where $Z_2 = \{0,1\}$ under two operations ' + ' and ' . ' can be defined as :

1.1 = 1.0 = 1 and 0.0 = 0.1 = 0

+	0	1
0	0	1
1	1	0

Table 2.1.

.	0	1
0	0	0
1	1	1

Table 2.2.

$(Z_2, +, .)$ is a near-ring but it is not a ring.

Definition 2.5.[3] A subgroup M of an near-ring N with $M.M \subseteq M$ is called a ***subnear-ring*** of N, (M ≤ N). A subgroup S of N with $NS \subseteq S$ is said to be a ***normal subgroup*** of N,$(S \trianglelefteq N)$.

Example 2.6.[5] let N={0,1,2,3,4,5}, definition of the addition and multiplication operation on N are in tables:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0

Table. 2.3

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	1	1	1	2	1	2
3	0	0	0	3	0	3
4	0	0	0	4	0	4
5	1	1	1	5	1	5

Table 2.4

(N , + , .) is near-ring. Let $S_1, S_2 \subset N$, $S_1=\{0,1\}$, $S_2=\{0,3,4\}$, therefore S_1 and S_2 are subnearring of N.

Definition 2.7. [6] Let H be a non-empty set, A ***hyperoperation*** ' $\circ$ ' on H is a map $\circ$: $H \times H \to P^*(H)$ Where $P^*(H)$ is the set of all non-empty subsets of H, usually, the **hyperoperation** is denoted by ' $\circ$ ' and the image of the pair (a , b) is denoted by " $a \circ b$ " and called the ***hyperproduct*** of a and b. If A and B are non-empty subsets of H .
Then $A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b$

Definition 2.8. [7] A ***hypergroupoid*** (H , $\circ$) is a non-empty set H together with a map $\circ$: $H \times H \to P^*(H)$, Where $P^*(H)$ denoted the set of all non-empty subsets of H. If A and B are non-empty subsets of H and
$x \in H$ Then

$$A \circ B = \bigcup_{\substack{a\in A \\ b\in B}} a \circ b,$$

$A \circ B$, $A \circ x$ and $x \circ B$ we mean: $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$.

Hypernear-rings generalize the concept of near-rings, in the sense that instead of the operation (+), the hyperopeartion (+) is define on the set N.

Definition 2.9.[8] the triple (N , + , .) is a ***hypernear-ring*** if:

I) (N,+) is a ***quasicanonical-hypergroup***. i.e. it satisfies the following axioms:
(1) $x + (y + z) = (x + y) + z$, for any $x, y, z \in N$.
(2) $\exists\, 0 \in N$ s.t. for any $x \in N$, $x + 0 = 0 + x = \{x\}$.
(3) for any $x \in N$, there exists a unique element $-x \in N$, such that

$$0 \in x + (-x) \cap (-x) + x.$$

(4) for any $x, y, z \in N$, $z \in x + y$ implies that

$$x \in z + (-y), \qquad y \in (-x) + z.$$

II) (N , $\cdot$) is a semigroup endowed with a two-sided absorbing element 0,
i.e . for any $x \in N$, $x \cdot 0 = 0 \cdot x = 0$.

III) The operation ' $\cdot$ ' is distributive with respect to the hyperoperation ' + ' from the left side: For many $x, y, z \in N$, $x \cdot (y + z) = x \cdot y + x \cdot z$.
If $x \in N$ and A, B are non- empty subsets of N,
Then by $A + B$, $A + x$ and $x + B$, we mean :

$$A + B = \bigcup_{\substack{a\in A \\ b\in B}} a + b$$

$A + x = A + \{x\}$, $x + B = \{x\} + B$, respectively.
Also, for all $x, y \in N$, we have $-(-x) = x$, $0 = -0$
Where 0 is unique and $-(x + y) = -y - x$.

Example 2.10.[2] Consider N={0,a,b,c} with ' + ' and ' $\cdot$ ' tables below:

+	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{a}	{0,a}	{b}	{c}
b	{b}	{b}	{0,a,c}	{b,c}
c	{c}	{c}	{b,c}	{0,a,b}

Table 2.5.

·	0	a	b	c
0	0	a	b	c
a	0	a	b	c
b	0	a	b	c
c	0	a	b	c

Table 2.6.

(N,+, ·) is a hypernear-ring.

3. Γ – hypernear-ring

Definition 3.1.[8] A ***Γ – hypernear-ring*** is a triple (M , + , Γ) where:
(1) Γ is a non-empty set of binary operations such that (M , +, α) is a hypernear-ring for each $\alpha \in \Gamma$.
(2) $x(y\beta z) = (x\alpha y)\beta z$ for all x,y,z ∈ M and $\alpha, \beta \in \Gamma$.

Example 3.2.[8] Let M={0,a,b} and Γ be the non-empty set of binary operation such that α , β ∈ Γ are defined as follows:

+	0	a	b
0	{0}	{a}	{b}
a	{a}	{0,a,b}	{a,b}
b	{b}	{a,b}	{0,a,b}

Table 3.1.

α	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

Table 3.2.

β	0	a	b
0	0	0	0
a	0	0	0
b	0	0	0

Table 3.3.

Then (M,+,Γ) is a Γ-hypernear-ring.

4. Hyper Ideals of Γ – hypernear-ring

Definition 4.1.[8] A subset I of a Γ- hypernear-ring M is called a ***left (resp. right) hyperideal*** of M if it satisfies:

- (I,+) is a normal subhypergroup of (M,+).
- $i\alpha u \in I$ (resp. $v\alpha(u + i) - u\alpha v \subseteq I$ for all i ∈ I, $\alpha \in \Gamma$ and u,v ∈M)

A subset I of M is called a hyperideal of M if it both a left hyperideal and right hyperideal .

Remarks 4.2. Let $(M, +, \Gamma)$ be a Γ hypernear − ring:
(1) The subset $M_0 = \{x \in M \mid 0\alpha x = 0\}$ of M is called a zero-symmetric part of M.
(2) If $M = M_0 \Longrightarrow$ M is zero-symmetric Γ- hypernear-ring.

Proposition 4.3. Let (M ,+, Γ) be a Γ-hypernear-ring, and if $\{A_k : k \in J\}$ is a family of hyperideals of M, then :

(1) $\bigcup_{k\in J} A_k$ is a hyperideal of M.

(2) $\bigcap_{k\in J} A_k$ is a hyperideal of M.

Proof.

(1) Since $0 \in \bigcup_{k\in J} A_k$, it follows that $\bigcup_{k\in J} A_k \neq \emptyset$. since $A_k \subseteq M$, for any $k \in J$, it follows that $\bigcup_{k\in J} A_k \subseteq M$. Let $a \in \bigcup_{k\in J} A_k$, then $a \in A_k$ for some $k \in J$, therefore $a + A_k = A_k + a = A_k$ (since A_k hyperideal), and

$$\bigcup_{k\in J}(a + A_k) = \bigcup_{k\in J}(A_k + a) = \bigcup_{k\in J} A_k \ ,$$

therefore $a + \bigcup_{k\in J} A_k = \bigcup_{k\in J} A_k + a = \bigcup_{k\in J} A_k$, hence $\bigcup_{k\in J} A_k$ is subhypergroup of M.

Now $a + A_k - a \subseteq A_k$ (since A_k hyperideal), then $\bigcup_{k\in J}(a + A_k - a) \subseteq \bigcup_{k\in J} A_k$, therefore $a + \bigcup_{k\in J} A_k - a \subseteq \bigcup_{k\in J} A_k$, Hence $\bigcup_{k\in J} A_k$ is normal subhypergroup.

let $\alpha \in \Gamma$ then $x\,\alpha\, u \in A_k$, $\forall\, x \in A_k$ *and* $u \in M$, (Since A_k is left hyperideal of M), therefore $x\,\alpha\, u \in \bigcup_{k\in J} A_k$, and

$v\,\alpha\,(u + x) - u\,\alpha\, v \subseteq A_k$, then $v\,\alpha\,(u + x) - u\,\alpha\, v \subseteq \bigcup_{k\in J} A_k$, Hence $\bigcup_{k\in J} A_k$ is hyperideal of M.

(2) Let $a \in \bigcap_{k\in J} A_k$, then $a \in A_k$ for all $k \in J$, therefore the prove is similar to (1).

Definition 4.4.[9] If M and M$'$ be two Γ- hypernear-ring the mapping φ:M →M$'$ is called a Γ- hypernear-ring homomorphism if for every $\alpha \in \Gamma$ we have:

- $\varphi(x + y) = \varphi(x) + \varphi(y)$
- $\varphi(x\alpha y) = \varphi(x)\alpha\varphi\, f(y)$

Clearly, a Γ- hypernear-ring homomorphism φ is an isomorphism if φ is injective and surjective. We write $M \cong M'$ If M isomorphic to M.

Definition 4.5[9] If I is a hyperideal of a hypernear-ring N, then we define the relation $x \equiv y \pmod{I} \Leftrightarrow (x - y) \cap I \neq \emptyset$. The relation is denoted by $x\, I^*\, y$. This is congruence relation on M.

The class x + I is represented by x and we denoted it with $I^*(x)$, Moreover, $I^*(x) = I^*(y) \Leftrightarrow x \equiv y (\mathrm{mod}\ I)$.

We can define M / I as follows: $M/I = \{I^* = x + I \mid x \in M\}$.

$I^*(x) \oplus I^*(y) = \{I^*(z) | z \in I^*(x) + I^*(y)\}$;

$I^*(x) \odot_\alpha I^*(y) = I^*(x\alpha y)$ for all $I^*(x), I^*(y) \in {}^{M}/_{I}$.

Theorem 4.6. (M/I, ⊕, ⊙) is a Γ-hypernear-ring.

Proof.

First: (M/I , ⊕ , ⊙) is a hypernear-ring was proved by :
V.Dasic,1991,Hypernear-Rings[10].

Second: $I^*(x) \odot_\alpha (I^*(y) \odot_\beta I^*(z)) = I^*(x) \odot_\alpha I^*(y\beta z) = I^*(x\alpha (y\beta z))$

Hence (M , + , Γ) is Γ-hypernear-ring and for all x,y,z ∈ M, $\alpha,\beta \in \Gamma$, therefore $(x\alpha (y\beta z)) = ((x\alpha y) \beta z)$, then

$I^*(x\alpha (y\beta z)) = I^*((x\alpha y) \beta z) = I^*(x\alpha y) \odot_\beta I^*(z)$

$= (I^*(x) \odot_\alpha I^*(y)) \odot_\beta I^*(z)$, for all $I^*(x), I^*(y), I^*(z) \in M/I$ and $\alpha,\beta \in \Gamma$.

Conclusion

- Arbitrary union and intersection of a family of hyperideals of Γ- hypernear-ring is hyperideal of Γ- hypernear-ring
- The quotient Γ-hypernear-ring (M/I , ⊕ , ⊙) is Γ-hypernear-ring, where M is Γ-hypernear-ring.

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