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Existence of solution of a coupled system of differential equation with the nonlocal two-point boundary conditions

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Existence of solution of a coupled system of differential equation with the nonlocal two-point boundary conditions

Abstract:

2 The aim of the paper is to study the existence of at least one solution of a nonlocal problem of coupled system of differential equations with nonlocal two-point boundary conditions. By means of Schauder fixed point theorem, an existence result of a solution is proved for the nonlocal problem considered.

Keywords: Coupled system, nonlocal conditions, at least one solution, Schauder fixed point theorem.

الملخص:

الهدف من البحث هو دراسة وجود حل واحد على الأقل لمشكلة غيرمحلية مكونة من نظام مزدوج من المعادلات التفاضلية مع شروط حدية ثنائية النقط غير محلية. باستخدام نظرية النقطة الثابتة لشودر تم أثبات وجود حل للمشكلة غير المحلية قيد الدراسة.

الكلمات المفتاحية: نظام مزدوج، شروط غير محلية، حل واحد على الأقل، نظرية النقطة الثابتة لشودر.

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I. Introduction

In the last decades or so, many authors have studied the existence of solutions for problems with nonlocal conditions. The reader is referred to ([2] –[13]) and references therein.

In [7] the authors studied the existence of at least one solution of the nonlocal problem

$$\frac{dx}{dt} = f_1(t, y(t)), \qquad t \in (0, T]$$
$$\frac{dy}{dt} = f_2(t, x(t)), \qquad t \in (0, T]$$

with the nonlocal conditions

$$x(0) + \sum_{k=1}^{n} a_k x(\tau_k) = x_0,\ a_k > 0, \tau_k \in (0, T]$$
$$y(0) + \sum_{j=1}^{m} b_j x(\eta_j) = y_0,\ b_j > 0, \eta_j \in (0, T].$$

In this system the unknown functions $x(t)$ and $y(t)$ depend only on t with two boundary conditions at multi points in the interval $(0, T]$, but in this paper we prove the existence of solutions for a nonlocal problem (1)-(4) in which the unknown functions y and x depend on functions φ_1 and φ_2 of t with two boundary conditions at two points in the interval $(0,1]$.

The nonlocal problem considered here is of the form

$$\frac{dx}{dt} = f_1(t, y(\varphi_1(t))), \qquad t \in (0,1] \qquad (1)$$
$$\frac{dy}{dt} = f_2(t, x(\varphi_2(t))), \qquad t \in (0,1] \qquad (2)$$

with the nonlocal two-point boundary conditions

$$x(\tau) + \alpha x(\eta) = 0, \qquad (3)$$
$$y(\tau) + \beta y(\eta) = 0, \qquad (4)$$

where $\tau \neq \eta, \tau, \eta \in [0,1]$ and $\alpha, \beta > 0$.

II. Preliminaries

In this section, review some definitions and theorems which will be need later.

Definition 2.1. [15] Let $F = \{f_i = X \to Y, i \in I\}$ be a family of functions with Y being a set of real (or complex) numbers, then we call uniformaly bounded if there exists a real number c such that $|f_i| \leq c \ \forall\ i \in I, x \in X$.

Definition 2.2. [15] Let $F = \{f(x)\}$ the class of functions defined on A where $A = [a, b] \subset R$, the class of functions $F = \{f(x)\}$ is equicontinuous if $\forall\ \epsilon > 0, \exists\ \delta(\epsilon)$ such that

$|x-y|<\delta$, implies that $|f(x)-f(y)|<\epsilon\ \forall f\in F,\ x,y\in A$.

Theorem 2.1. [1] The function $f(x)=(f_1(x),f_2(x),\cdots,f_n(x))$ is uniformly continuous in $I=[a,b]$ if and only if each f_i is uniformly continuous in I.

Theorem 2.2. [15] (Lebesgue Dominated Convergence Theorem)

Let f_n be a sequence of functions converging to a limit f of A, and suppose that $|f_n(t)|\le \phi(t), t\in A, n=1,2,3,\dots$ were ϕ is integrable on A. Then

1. f is integrable on A
2. $\lim_{n\to\infty}\int_A f_n(t)d\mu = \int_A f(t)d\mu$.

Theorem 2.3. [14] (Schauder theorem)

Let Q be a convex subset of a Banach space $X, F: Q \to Q$ be a compact and continuous map, then F has at least one fixed point in Q.

III. Existence theorem

In this section we will study the existence of at least one solution of nonlocal problem (1)-(4).

Let X be the class of all ordered pairs (x,y), $x,y\in C(0,1]$ with the norm

$$\|(x,y)\|=\|x\|+\|y\|=sup_{t\in[0,1]}|x(t)|+sup_{t\in[0,1]}|y(t)|,$$

and let $\varphi_i:[0,1]\to[0,1]$ be continuous function for $i=1,2$.

The nonlocal problem(1)-(4) will be considered under the following assumptions:

(h_1) $f_i:[0,1]\times R\to R$ satisfies Caratheodory conditions, that is f_i is

(i) measurable in $t\in(0,1]$, for any $x\in R$.

(ii) continuous in $x\in R$, for almost all $t\in(0,1]$.

(h_2) There exist two integrable functions $a_i\in L_1[0,1]$ and two positive constants $b_i>0$, $i=1,2$ such that

$$|f_i(t,x)|\le a_i(t)+b_i|x|,$$

$$sup_{t\in[0,1]}\int_0^t a_i(s)\,ds<a_i^*,\quad i=1,2\ \forall\, t\in[0,1]$$

where a_i^* for $i=1,2$ are finite real numbers, and $L_1[0,1]$ denotes the class of Lebesgue integrable functions on the interval $I=[0,1]$.

Then we have the following lemma:

Lemma 3.1 The solution of the nonlocal problem (1)-(4) can be expressed by system of the integral equations

$$x(t)=\int_0^t f_1\left(s,y(\varphi_1(s))\right)ds-\frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s,y(\varphi_1(s))\right)ds-$$

$$-\frac{1}{1+\alpha}\int_0^\tau f_1\left(s,y(\varphi_1(s))\right)ds, \tag{5}$$

and

$$y(t)=\int_0^t f_2\left(s,x(\varphi_2(s))\right)ds-\frac{\beta}{1+\beta}\int_0^\eta f_2\left(s,x(\varphi_2(s))\right)ds-$$

$$-\frac{1}{1+\beta}\int_0^\tau f_2\left(s,x(\varphi_2(s))\right)ds. \tag{6}$$

Proof. Integrating equation (1), we obtain

$$x(t)=x(0)+\int_0^t f_1\left(s,y(\varphi_1(s))\right)ds,$$

$$x(\eta)=x(0)+\int_0^\eta f_1\left(s,y(\varphi_1(s))\right)ds$$

and

$$\alpha x(\eta)=\alpha x(0)+\alpha\int_0^\eta f_1\left(s,y(\varphi_1(s))\right)ds \tag{7}$$

Let $t=\tau$, then

$$x(\tau)=x(0)+\int_0^\tau f_1\left(s,y(\varphi_1(s))\right)ds. \tag{8}$$

Substituting from equations (7) and (8), we get

$$x(0)=-\frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s,y(\varphi_1(s))\right)ds-\frac{1}{1+\alpha}\int_0^\tau f_1\left(s,y(\varphi_1(s))\right)ds.$$

Hence

$$x(t)=\int_0^t f_1\left(s,y(\varphi_1(s))\right)ds-\frac{\alpha}{1+\alpha}\int_0^\eta f_1\left(s,y(\varphi_1(s))\right)ds-$$

$$-\frac{1}{1+\alpha}\int_0^\tau f_1\left(s,y(\varphi_1(s))\right)ds.$$

Similarly, Integrating equation (2), we obtain

$$y(t)=y(0)+\int_0^t f_2\left(s,x(\varphi_2(s))\right)ds,$$

$$y(\eta)=y(0)+\int_0^\eta f_2\left(s,x(\varphi_2(s))\right)ds,$$

and

$$\beta x(\eta) = \beta x(0) + \beta \int_0^{\eta} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds. \tag{9}$$

Let $t = \tau$, then

$$y(\tau) = y(0) + \int_0^{\tau} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds. \tag{10}$$

Substituting from equations (9) and (10), we get

$$y(0) = -\frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds .$$

Hence

$$y(t) = \int_0^{t} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds.$$

Now, differentiating (5) and (6), we obtain

$$\frac{dx}{dt} = \frac{d}{dt}\int_0^{t} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds + 0.$$

This mean that

$$\frac{dx}{dt} = f_1\left(t, y\left(\varphi_1(t)\right)\right).$$

Similarly,

$$\frac{dy}{dt} = \frac{d}{dt}\int_0^{t} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds + 0.$$

This mean that

$$\frac{dy}{dt} = f_2\left(t, x\left(\varphi_2(t)\right)\right).$$

Also, from (5) and (6), we get

$$x(\tau) = \int_0^{\tau} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds$$
$$= \frac{\alpha}{1+\alpha}\left(\int_0^{\tau} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \int_0^{\eta} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds\right),$$

and

$$x(\eta) = \int_0^{\eta} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds$$
$$= \frac{\alpha}{1+\alpha}\left(\int_0^{\eta} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds - \int_0^{\tau} f_1\left(s, y\left(\varphi_1(s)\right)\right) ds\right).$$

Hence

$$x(\tau) + \alpha x(\eta) = 0.$$

Similarly

$$y(\tau) = \int_0^{\tau} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x\left(\varphi_2(s)\right)\right) ds.$$

$$= \frac{\beta}{1+\beta}\left(\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds\right)$$

and

$$y(\eta) = \int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds.$$

$$= \frac{\beta}{1+\beta}\left(\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds\right).$$

Hence

$$y(\tau) + \beta y(\eta) = 0.$$

This proves the equivalence between the problem (1) – (4) and the integral equations (5) and (6).

Now we are prepared to formulate our main existence result.

Theorem 3. 1 Let the assumptions (h_1) - (h_2) are satisfied. Then there exist at least one solution of the nonlocal problem (1) – (4).

Proof. Consider the operator $\mathcal{F}$ by

$$\mathcal{F}(x, y) = (\mathcal{F}_1 y, \mathcal{F}_2 x),$$

were

$$\mathcal{F}_1 y = \int_0^{t} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds,$$

$$\mathcal{F}_2 x = \int_0^{t} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds.$$

Define the set Q_r as follows:

$$Q_r = \left\{(x, y) : \|(x, y)\| = \|x\| + \|y\| \le r, r = r_1 + r_2, r_1 = \frac{2a_1^*}{1-3b_1},\ r_2 = \frac{2a_2^*}{1-3b_2}\right\}$$

Now

$$|\mathcal{F}_1 y| = \left|\int_0^{t} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \right.$$

$$\left. - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds\right| \le \left|\int_0^{t} f_1\left(s, y(\varphi_1(s))\right) ds\right| + \frac{\alpha}{1+\alpha}\left|\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds\right| +$$

$$+ \frac{1}{1+\alpha}\left|\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds\right| \le \int_0^{t}\left|f_1\left(s, y(\varphi_1(s))\right)\right| ds + \frac{\alpha}{1+\alpha}\int_0^{\eta}\left|f_1\left(s, y(\varphi_1(s))\right)\right| ds +$$

$$+ \frac{1}{1+\alpha}\int_0^{\tau}\left|f_1\left(s, y(\varphi_1(s))\right)\right| ds \le$$

$$\int_0^{t} a_1(s) ds + b_1\int_0^{t}|y(\varphi_1(s))| ds + \frac{\alpha}{1+\alpha}\left(\int_0^{\eta} a_1(s) ds + b_1\int_0^{\eta}|y(\varphi_1(s))| ds\right) +$$

$$\frac{1}{1+\alpha}\left(\int_0^{\tau} a_1(s)ds + b_1 \int_0^{\tau} |y(\varphi_1(s))|ds\right) \leq$$

$$a_1^* + b_1 r_1 + \frac{\alpha}{1+\alpha}(a_1^* + b_1 r_1) + \frac{1}{1+\alpha}\,(a_1^* + b_1 r_1) \leq r_1.$$

Similarly, we have

$$|\mathcal{F}_2 x| = \left|\int_0^t f_2\left(s, x(\varphi_2(s))\right) ds - \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \right.$$

$$\left. - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds\right| \leq \left|\int_0^t f_2\left(s, x(\varphi_2(s))\right) ds\right| + \frac{\beta}{1+\beta}\left|\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds\right| +$$

$$+ \frac{1}{1+\beta}\left|\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds\right| \leq \int_0^t \left|f_2\left(s, x(\varphi_2(s))\right)\right| ds + \frac{\beta}{1+\beta}\int_0^{\eta} \left|f_2\left(s, x(\varphi_2(s))\right)\right| ds +$$

$$+ \frac{1}{1+\beta}\int_0^{\tau} \left|f_2\left(s, x(\varphi_2(s))\right)\right| ds$$

$$\leq \int_0^t a_2(s)ds + b_2 \int_0^t |x(\varphi_2(s))|ds + \frac{\beta}{1+\beta}\left(\int_0^{\eta} a_2(s)ds + + b_2 \int_0^{\eta} |x(\varphi_2(s))|ds\right) +$$

$$\frac{1}{1+\beta}\left(\int_0^{\tau} a_2(s)ds + b_2 \int_0^{\tau} |x(\varphi_2(s))|ds\right) \leq$$

$$a_2^* + b_2 r_2 + \frac{\beta}{1+\beta}(a_2^* + b_2 r_2) + \frac{1}{1+\beta}\,(a_2^* + b_2 r_2) \leq r_2.$$

Hence

$$\|\mathcal{F}(x,y)\| = \|\,\mathcal{F}_1 y\| + \|\mathcal{F}_2 x\| \leq r_1 + r_2 = r,$$

then the class of functions $\{\mathcal{F}(x,y)\}$ is uniformly bounded.

Let $(x,y) \in Q_r$, for $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $and \;\; let \; |t_2 - t_1| < \delta$, then

$$|\,\mathcal{F}_1 y(t_2) - \mathcal{F}_1 y(t_1)| = \left|\int_0^{t_2} f_1\left(s, y(\varphi_1(s))\right) ds - \right.$$

$$- \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds - \frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds -$$

$$\int_0^{t_1} f_1\left(s, y(\varphi_1(s))\right) ds + + \frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s, y(\varphi_1(s))\right) ds +$$

$$\left.\frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s, y(\varphi_1(s))\right) ds\right| \leq \int_{t_1}^{t_2} \left|f_1\left(s, y(\varphi_1(s))\right)\right| ds \leq$$

$$\leq \int_{t_1}^{t_2} (|a_1(s)| + b_1 |y(\varphi_1(s))|) ds \leq \int_{t_1}^{t_2} a_1(s)ds + b\,\|y\|\,(t_2 - t_1),$$

then $\{\,\mathcal{F}_1 y\}$ is a class of equicontinuous functions.

Similarly, we have

$$|\,\mathcal{F}_1 x(t_2) - \mathcal{F}_1 x(t_1)| = \left|\int_0^{t_2} f_2\left(s, x(\varphi_2(s))\right) ds - \right.$$

$$- \frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s, x(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^{\tau} f_2\left(s, x(\varphi_2(s))\right) ds - \int_0^{t_1} f_2\left(s, x(\varphi_2(s))\right) ds +$$

$$+\frac{\beta}{1+\beta}\int_0^{\eta} f_2\left(s,x(\varphi_2(s))\right)ds+\frac{1}{1+\beta}\int_0^{\tau} f_2\left(s,x(\varphi_2(s))\right)ds\Big|\le\int_{t_1}^{t_2}\left|f_2\left(s,x(\varphi_2(s))\right)\right|ds\le$$

$$\le\int_{t_1}^{t_2}(|a_2(s)|+b_2|x(\varphi_2(s))|)ds\le\int_{t_1}^{t_2}a_2(s)ds+b\,\|x\|\,(t_2-t_1),$$

then $\{\mathcal{F}_1 x\}$ is a class of equicontinuous functions.

Therefore the class of functions $\{\mathcal{F}(x,y)\}$ is equicontinuous and uniformly bounded.

Let $\{y_n\}$ be convergent sequence such that $y_n\to y$, then

$$lim_{n\to\infty}\ \mathcal{F}_1(y_n(\varphi_1(s)))=lim_{n\to\infty}\int_0^t f_1\left(s,y_n(\varphi_1(s))\right)ds-$$

$$-\frac{\alpha}{1+\alpha}\int_0^{\eta} f_1\left(s,y_n(\varphi_1(s))\right)ds-\frac{1}{1+\alpha}\int_0^{\tau} f_1\left(s,y_n(\varphi_1(s))\right)ds.$$

But from the assumption

$$\left|f_1\left(s,y_n(\varphi_1(s))\right)\right|\le a_1+\ b_1|y_n|$$

$$\le a_1+b_1 r,$$

$$f_1\left(s,y_n(\varphi_1(s))\right)\to f_1\left(s,y(\varphi_1(s))\right).$$

Applying Lebesgue dominated convergence theorem, we deduce that

$$lim_{n\to\infty}\int_0^t f_1\left(s,y_n(\varphi_1(s))\right)ds=\int_0^t \lim_{n\to\infty} f_1\left(s,y_n(\varphi_1(s))\right)ds$$

$$=\int_0^t f_1(s,\lim_{n\to\infty} y_n(\varphi_1(s)))\,ds=\int_0^t f_1\left(s,y(\varphi_1(s))\right)ds,$$

$$lim_{n\to\infty}\int_0^{\eta} f_1\left(s,y_n(\varphi_1(s))\right)ds=\int_0^{\eta} \lim_{n\to\infty} f_1\left(s,y_n(\varphi_1(s))\right)ds$$

$$=\int_0^{\eta} f_1(s,\lim_{n\to\infty} y_n(\varphi_1(s)))\,ds=\int_0^{\eta} f_1\left(s,y(\varphi_1(s))\right)ds,$$

and

$$lim_{n\to\infty}\int_0^{\tau} f_1\left(s,y_n(\varphi_1(s))\right)ds=\int_0^{\tau} \lim_{n\to\infty} f_1\left(s,y_n(\varphi_1(s))\right)ds$$

$$=\int_0^{\tau} f_1(s,\lim_{n\to\infty} y_n(\varphi_1(s)))\,ds=\int_0^{\tau} f_1\left(s,y(\varphi_1(s))\right)ds,$$

then

$$lim_{n\to\infty}\ \mathcal{F}_1(y_n(\varphi_1(s)))=\int_0^t f_1\left(s,y_n(\varphi_1(s))\right)ds-\int_0^{\eta} f_1\left(s,y_n(\varphi_1(s))\right)ds-$$

$$-\int_0^{\tau} f_1\left(s,y_n(\varphi_1(s))\right)ds=\mathcal{F}_1 y$$

which proves that $\mathcal{F}_1$ is continuous operator.

Also

Let $\{x_n\}$ be convergent sequence such that $x_n \to x$, then

$$lim_{n\to\infty} \mathcal{F}_2(x_n(\varphi_2(s))) = lim_{n\to\infty} \int_0^t f_2\left(s, x_n(\varphi_2(s))\right) ds -$$

$$-\frac{\beta}{1+\beta}\int_0^\eta f_2\left(s, x_n(\varphi_2(s))\right) ds - \frac{1}{1+\beta}\int_0^\tau f_2\left(s, x_n(\varphi_2(s))\right) ds.$$

But from the assumption

$$\left|f_2\left(s, x_n(\varphi_2(s))\right)\right| \le a_2 + b_2|x_n|$$

$$\le a_2 + b_2 r,$$

$$f_2\left(s, x_n(\varphi_2(s))\right) \to f_2\left(s, x(\varphi_2(s))\right).$$

Applying Lebesgue dominated convergence theorem, we deduce that

$$lim_{n\to\infty} \int_0^t f_2\left(s, x_n(\varphi_2(s))\right) ds = \int_0^t \lim_{n\to\infty} f_2\left(s, x_n(\varphi_2(s))\right) ds$$

$$= \int_0^t f_2(s, \lim_{n\to\infty} x_n(\varphi_2(s))) ds = \int_0^t f_2\left(s, x(\varphi_2(s))\right) ds,$$

$$lim_{n\to\infty} \int_0^\eta f_2\left(s, x_n(\varphi_2(s))\right) ds = \int_0^\eta \lim_{n\to\infty} f_2\left(s, x_n(\varphi_2(s))\right) ds$$

$$= \int_0^\eta f_2(s, \lim_{n\to\infty} x_n(\varphi_2(s))) ds = \int_0^\eta f_2\left(s, x(\varphi_2(s))\right) ds,$$

and

$$lim_{n\to\infty} \int_0^\tau f_2\left(s, x_n(\varphi_2(s))\right) ds = \int_0^\tau \lim_{n\to\infty} f_2\left(s, x_n(\varphi_2(s))\right) ds$$

$$= \int_0^\tau f_2(s, \lim_{n\to\infty} x_n(\varphi_2(s))) ds = \int_0^\tau f_2\left(s, x(\varphi_2(s))\right) ds,$$

then

$$lim_{n\to\infty} \mathcal{F}_2(x_n(\varphi_2(s))) = \int_0^t f_2\left(s, x_n(\varphi_2(s))\right) ds - \int_0^\eta f_2\left(s, x_n(\varphi_2(s))\right) ds -$$

$$-\int_0^\tau f_2\left(s, x_n(\varphi_2(s))\right) ds = \mathcal{F}_2 x$$

which proves that $\mathcal{F}_2$ is continuous operator.

Hence, $\mathcal{F}: Q_r \to Q_r$ is continuous and compact.

Now we show that Q_r is convex,

let $(x_1, y_1), (x_2, y_2) \in Q_r$,

$$\|(x_i, y_i)\| = \|x_i\| + \|y_i\| \le r, i = 1,2.$$

For $\lambda \in [0,1]$,

$$\|\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)\| = \|(\lambda x_1, \lambda y_1) + ((1-\lambda)x_2, (1-\lambda)y_2)\|$$

$$= \|(\lambda x_1 + (1-\lambda)x_2,\ \lambda y_1 + (1-\lambda)y_2)\|$$
$$\le \|\lambda x_1 + (1-\lambda)x_2\| + \|\lambda y_1 + (1-\lambda)y_2\|$$
$$\le \lambda\|x_1\| + (1-\lambda)\|x_2\| + \lambda\|y_1\| + (1-\lambda)\|y_2\|$$
$$= \lambda[\|x_1\| + \|y_1\|] + (1-\lambda)[\|x_2\| + \|y_2\|]$$
$$\le \lambda r + (1-\lambda)r = r,$$

this mean that Q_r is convex.

Using Schauder fixed point theorem, $\mathcal{F}$ has a fixed point $(x, y) \in Q_r$ which proves that there exists at least one solution of the nonlocal problem (1) – (4).

This completes the proof.

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