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The generalized Gamma Matrix Function via Jordan Canonical Form and its Approximations

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The generalized Gamma Matrix Function via Jordan Canonical Form and its Approximations

Abstract

The generalized Gamma matrix function via Jordan canonical form are provided. A more general case of Beta matrix functions with two positive stable matrices are also obtained. Asymptotic approximations are derived for the Gamma matrix function with two positive stable matrices.

Keywords: Special functions; Asymptotic Approximations; Matrix; Gamma function of Matrices; Jordan Canonical Form.

AMS Subject Classification: 33C05 , 33C15, 34A05, 15A60, 41A30,41A60.

الملخص

استنتجنا صيغة معممة لدالة مصفوفة جاما عن طريق الشكل المعيارى لجوردان. ولكل مصفوفتين لهما قيم ذاتية موجبة عممنا مصفوفة بيتا وتم تقدير مقارب لدالة مصفوفة جاما .

I. introduction

The Gamma function , which is defined by the convergent improper integral

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt \qquad Re(\lambda) \geq 0 \qquad (1.1)$$

see e.g. E. F. Rainville [22], G. Andrews et al [2] and F. W. J. Olver [21], has been an important tool in numerous branches of mathematical analysis and applications. In the past two decades generalization and extensions of scalar special functions to Matrix special functions have been developed. The Gamma matrix function, whose eigenvalues are all in the right open half- plane isintroduced and studied in L. J_odar, J. Cort_es [13] for matrices in $\mathbb{C}^{r\times r}$. Hermite matrix polynomials are introduced by L. J_odar et al [12] and some of their properties are given in E. Defez, L. J_odar [4]. Other classical orthogonal polynomials as Laguerre and Chebyshev have been extended to orthogonal matrix polynomials, and some results have been investigated in L. J_odar, J. Sastre [15] and E. Defez, L. J_odar [5]. Relations between the Beta, Gamma and the Hypergeometric matrix function are given in L.

J_odar, J. G. Cort_es [14] and R. S. Batahan [3]. The gamma function, the extended gamma function, the beta function, the extended beta function, the gamma distribution, the beta distribution and the extended beta distribution have been generalized to the matrix case in various ways. These generalizations and some of their properties can be found in Gupta and Nagar [9]Nagar, Gupta, and Sánchez [16], Nagar, Roldán-Correa and Gupta [17], Nagar and Roldán- Correa [18], and Nagar, Morán-Vásquez and Gupta [19]. For some recent advances the reader is refereed to Hassairi and Regaig [10].Some integrals involving zonal polynomials and generalized extended matrix variate beta function are evaluated see [20]. These special functions of matrices have become an important tool in both theory and applications. The order of presentation in this article is as follows. In section 2 we provide basic necessary notation, definitions and auxiliary theorems that need to be cited in the sequel. In section 3 We consider a more general case of Gamma and Beta matrix functions with two positive stable matrices.

II. Preliminaries

In this section we elaborate on some necessary language that is adopted from L. Jodar, J. Sastre [13] and N. J. Higham [11]. We also record some basic theorems from asymptotic analysis that can be found in e.g. W. Wasow [23] and A. Erdelyi [7] and that will be needed in proving our main results.

Denote by $\lambda_1,\cdots\lambda_n$ the distinct eigenvalues of a matrix $P \in \mathbb{C}^{r\times r}$. The spectrum $\sigma(P)$ of$P \in \mathbb{C}^{r\times r}$, denotes the set of all the eigenvalues of P. The 2-norm of P will be denoted by $\|P\|$ and it is defned by $\|P\| = \begin{matrix} sup \\ x \neq 1 \end{matrix} \frac{\|P_x\|_2}{\|x\|_2}$, where for a y in $\mathbb{C}^{r\times r}$, $\|y\|_2 = \langle y^H, y\rangle^{\frac{1}{2}}$ is the Euclidean norm of y, and y^H denotes the Hermitian adjoint of y. We put $\zeta(P)$ and $\varrho(P)$ the real numbers

$$\gamma(P) = max\{Re(\lambda) : \lambda \in \sigma(P)\}, \varrho(P) = min\{Re(\lambda) : \lambda \in \sigma(P)\} \qquad (2.1)$$

A holomorphic function $f(\lambda)$ at a point was defined as a regular analytic function in a neighborhood of the point, see e.g. W. Wasow [23]. It is called holomorphic in a set if it is holomorphic at every point of the set. A matrix is called holomorphic if every entry of it is a holomorphic function.

We now give a definition and elementary properties of asymptotic series see e.g. W. Wasow [23] and A. Erdelyi [7]. If $f(\lambda)$ and $g(\lambda)$ are homomorphic functions of the complex variable λ, which are defined in an open set of the complex plane, and P is matrix in $\mathbb{C}^{r\times r}$ with $\sigma(P) \subset \Omega$, then from the properties of the matrix functional calculus, see N. Dunford, J. Schwartz [6], it follows that

$$f(P)g(P) = g(P)f(P) \qquad (2.2).$$

Definition 2.1

A set of complex numbers is called positive stable if all the elements of the set have positive real part and a square matrix P is called positive stable if $\sigma(P)$ is positive stable.

If P is a positive stable matrix in $\mathbb{C}^{r\times r}$, than $\Gamma(P)$ is well defined, see L. Jodar, J. G. Cortes [13]

$$\Gamma(P) = \int_0^\infty e^{-t} t^{P-I} dt, \qquad t^{P-I} = exp\big((P-I)lnt\big) \qquad (2.3)$$

The reciprocal Gamma function denoted by $\Gamma^{-1} = \frac{1}{\Gamma}$, is an entire function of the complex variable λ. Then the image of $\Gamma^{-1}(\lambda)$, for any P in $\mathbb{C}^{r\times r}$, the Riesz Dunford functional calculus shows that the image of $\Gamma^{-1}(\lambda)$ acting on P, denoted by $\Gamma^{-1}(P)$ is well defined. See N. Dunford, J. Schwartz [6].

Furthermore, if

$$P + nI \text{ is invertible for every integer } n \geq 0 \qquad (2.4)$$

then $\Gamma(P)$ is invertible, its inverse coincides with $\Gamma^{-1}(\lambda)$, and one gets the formula

$$P(P+I)\dots(P+(n-1))I\,\Gamma^{-1}(P+nI) = \Gamma^{-1}(P)\;, n \geq 1 \qquad (2.5)$$

Under condition(2.4), by (2.2), equation (2.5) can be written in the form

$$P(P+I)\dots(P+(n-1))I = \Gamma(P+nI)\Gamma^{-1}(P)\;, n \geq 1 \qquad (2.6)$$

If we take into account the scalar factorial function denoted by $(\lambda)_n$ and defined by $(\lambda)_n = \lambda(\lambda+1)\dots(\lambda+n-1), n \geq 1, (\lambda)_0 = 1;$ then by application of the matrix functional calculus to this function, for any matrix P in $\mathbb{C}^{r\times r}$ on gets

$$(P)_n = P(P+I)\dots(P+n-I), n \geq 1, P_0 = I \qquad (2.7)$$

If $f(P)$ is well defined and T is an invertible matrix in $\mathbb{C}^{r\times r}$, then

$$f(TPT^{-1}) = Tf(P)T^{-1} \qquad (2.8)$$

$$\|e^{-tP}\| \le e^{-t\varrho(P)} \sum_{k=0}^{r-1} \frac{\left(\|P\| r^{\frac{1}{2}}\right)^k}{k!} \quad t \ge 0 \qquad (2.9)$$

In particular, if $t \ge 1$, $t^P = e^{P \ln t}$ satisfies

$$\|t^P\| \le t^{\gamma(P)} \sum_{k=0}^{r-1} \frac{\left(\|P\| r^{\frac{1}{2}} \ln t\right)^k}{k!} \quad t \ge 0 \qquad (2.10)$$

It is a standard result that for any matrix $P \in \mathbb{C}^{r\times r}$ there exist a nonsingular matrix $T \in \mathbb{C}^{r\times r}$ (T dependent on eigenvalues or it is a constant matrix) such that

$$T^{-1}PT = J = diag(J_1, J_2, \dots, J_s) \qquad (2.11)$$

Where

$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda_k & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_k \end{bmatrix} \in \mathbb{C}^{m_k \times m_k} \qquad (2.12)$$

or

$J_k(\lambda) = [\lambda_k] \in \mathbb{C}^{1\times 1}$

where $m_1 + m_2 + \cdots + m_s = r$. We can write a Jordan block $J_k(\lambda)$ as

$$J_k(\lambda) = \lambda_k I_{m_k} + H_{m_k} \qquad (2.13)$$

where I_{m_k} is an identity matrix of size $m_k \times m_k$ and,

$$\mathrm{H}_{\mathrm{m_k}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \text{ of size } \mathrm{m_k} \times \mathrm{m_k} \qquad (2.14)$$

Definition 2.2

The function f is said to be defined on $\sigma(P)$ if the values

$$f^{(j)}(\lambda_i), \quad 0 \le j \le r_i - 1 , 1 \le i \le n$$

exist. These are called the values of the function f on $\sigma(P)$.

The following of $f(P)$ requires only the values of f on $\sigma(P)$, it does not require any other information about f see N. J. Higham [11]. Ti is well knows that if $f(P)$ is well defined and T is an invertible matrix in $\mathbb{C}^{r\times r}$, then

$$f(T^{-1}PT) = T^{-1}f(P)T \tag{2.15}$$

The symbols O, o and $\sim$, due to Bachmann and Landau (1927), which are also used by e.g. F. W. J. Olver [21] and A. Erd´elyi [7]. Concerning the definition and elementary properties of asymptotic series we refer to W. Wasow [23] and A. Erd´elyi [7].

Lemma 2.1 (matrix function via Jordan canonical form). Let f be defined on $\sigma(P)$,$P \in \mathbb{C}^{r\times r}$ and let P have the Jordan canonical form (2.11) subject to (2.12). Then

$$f(P) = Tf(J)T^{-1} = Tdiag\big(f(J_1), f(j_2), \cdots, f(J_s)\big)T^{-1} \tag{2.16}$$

where

$$f(J_k) = \begin{bmatrix} f(\lambda_k) & f^{(1)}(\lambda_k) & \cdots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ 0 & f(\lambda_k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & f^{(1)}(\lambda_k) \\ 0 & \cdots & 0 & f(\lambda_k) \end{bmatrix} \in \mathbb{C}^{m_k\times m_k} \tag{2.17}$$

Proof See [11].

III. More general case of Gamma and Beta matrix functions with two positive stable matrices.

It is possible to extend the classical Gamma function in many ways, some of these extensions could be useful in certain types of problems. In their work on the subject, M. Abul-Dahab and A. Bakher see [1] define the Gamma matrix functions as follows.

$$\Gamma(A,B) = \int_0^{\infty} t^{P-I} e^{It+\frac{B}{t}}\, dt \tag{3.1}$$

where A and B are a positive stable matrices and I ia an identity. In this paper we define the Gamma matrix function with two positive stable matrices as follows.

Definition 3.1 Let P and Q be two positive stable matrices in$\mathbb{C}^{r\times r}$, then the generalized Gamma matrix functions denoted by $\Gamma(Q,P)$ and define as

$$\Gamma(Q,P) = \int_0^{\infty} e^{-tQ} t^{P-I}\, dt \tag{3.2}$$

The definition of generalized Gamma function, $\Gamma(Q,P)$ is claimed well defined, indeed,

$$\|\Gamma(Q,P)\| = \left\| \int_0^{\infty} e^{-tQ} t^{P-I}\, dt \right\|$$

$$\leq \int_0^{\infty} \|e^{-Qt}\|\, \|t^{P-I}\| dt \tag{3.3}$$

using equations (2.9) and(2.10) we have

$$\|\Gamma(Q,P)\| \leq \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|)^k r^{\frac{j+k}{2}}}{j!\,k!}\int_0^\infty e^{-tvarrho(Q)}t^{\gamma(P)-1}ln(t)^j(t)^k dt$$

$$\leq \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|)^k r^{\frac{j+k}{2}}}{j!\,k!}\int_0^\infty e^{-t\varrho(Q)}t^{\gamma(P)-1}ln(t)^j(t)^k dt$$

$$\leq \sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|)^k r^{\frac{j+k}{2}}}{j!\,k!}\int_0^\infty e^{-t\varrho(Q)}t^{\gamma(P)+j+k-1}dt$$

let $\tau = t\varrho(Q)$, $dt = \frac{d\tau}{\varrho(Q)}$ therefore,

$$\sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|)^k r^{\frac{j+k}{2}}}{j!\,k!}(\varrho(Q))^{-\gamma(P)-j-k}\int_0^\infty e^{-\tau}\tau^{\gamma(P)+j+k-1}d\tau =$$

$$\sum_{j=0}^{r-1}\sum_{k=0}^{r-1}\frac{(\|P\|+1)^j(\|Q\|)^k r^{\frac{j+k}{2}}}{j!\,k!}(\varrho(Q))^{-\gamma(P)-j-k}\Gamma(\gamma(P)-j-k) < \infty$$

That is

$$\|\Gamma(Q,P)\| < \infty$$

From the definition (5.1) we see that

$$\Gamma(Q,P+I) = \int_0^\infty e^{-Qt}t^P dt$$

$$= -Q^{-1}e^{-Qt}t^P|_0^\infty + Q^{-1}P\int_0^\infty e^{-Qt}t^{P-I}dt$$

$$= 0 + Q^{-1}P\Gamma(Q,P)$$

Thus we have

$$\Gamma(Q,P+I) = Q^{-1}P\Gamma(Q,P)$$

In addition $\Gamma(Q,I) = Q^{-1}$ which is easily derived from the definition.

We then have the identity

$$\Gamma(Q,P+nI) = Q^{-1}(P+(n-1)I)\Gamma(Q,P+(n-1)I)$$

$$= (Q^{-1})^2(P+(n-1)I)(P+(n-2)I)\Gamma(Q,P+(n-2)I)$$

$$=\vdots$$

$$=(Q^{-1})^n(P+(n-1)I)(P+(n-2)I)\cdots\Gamma(Q,P)$$

$$=(Q^{-1})^n(P)_n\Gamma(Q,P)$$

where $(P)_n=(P+(n-1)I)(P+(n-2)I)\cdots P$.

Lemma 3.1 Let P and Q be two positive stable matrices in $\mathbb{C}^{r\times r}$ and suppose also that P and Q commute and there exist a nonsingular matrix T satisfying

$$T^{-1}PT=J_1$$

and

$$T^{-1}QT=J_2$$

where J_1 and J_2 are Jordan matrices

$$J_1=\begin{bmatrix} J_{1_1} & 0 & \cdots & \cdots & 0 \\ 0 & J_{1_2} & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J_{1_s} \end{bmatrix}$$

$$J_2=\begin{bmatrix} J_{2_1} & 0 & \cdots & \cdots & 0 \\ 0 & J_{2_2} & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & J_{2_s} \end{bmatrix}$$

such that J_{1_i} has the same size as the size of J_{2_i} for all $i=j,\ 1\le i,j\le s$, then we can define the generalized Gamma function $\Gamma(Q,P)$ as

$$\Gamma(Q,P)=T^{-1}\Gamma(J_1,J_2)T \qquad (3.4)$$

Proof:

$$\Gamma(Q,P)=\int_0^\infty e^{-Qt}t^{P-I}dt,$$

by equation (2.15) we have,

$$\Gamma(Q,P)=\int_0^\infty T^{-1}e^{-tJ_2}T\ \ T^{-1}t^{J_1-I}T\,dt$$

$$=T^{-1}\int_0^\infty e^{-tJ_2}t^{J_1-I}dt\,T$$

$$=T^{-1}\Gamma(J_2,J_1)T. \quad \square$$

Lemma 3.2 (generalized Gamma Matrix Function via Jordan Canonical Form)

Let P and Q have the Jordan canonical forms

$$T^{-1}PT = J_1 = diag(J_{1_1}, J_{1_2} \cdots J_{1_s})$$

$$T^{-1}PT = J_2 = diag(J_{2_1}, J_{2_2} \cdots J_{2_s})$$

J_{1_i} have the same size as the size of J_{2_j} for all $i = j, 1 \le i, j \le s$

$$J_{1_k} = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_k \end{bmatrix} \in \mathbb{C}^{r_k \times r_k}$$

$$J_{2_k} = \begin{bmatrix} \mu_k & 1 & 0 & \cdots & 0 \\ 0 & \mu_k & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \mu_k \end{bmatrix} \in \mathbb{C}^{r_k \times r_k}$$

$$\Gamma(J_{1_k}, J_{2_k}) = \int_0^\infty e^{-J_{2_k} t} t^{J_{1_k} - I} dt \tag{3.5}$$

$$= \begin{bmatrix} \Gamma(\lambda_k, \mu_k) & \frac{\partial \Gamma(\lambda_k,\mu_k)}{\partial \lambda_k} + \frac{\partial \Gamma(\lambda_k,\mu_k)}{\partial \mu_k} & \cdots & \sum_{j=0}^{r_k-1} \frac{\partial^j \partial^{(r_k-1)-j}}{\partial \mu_k^j \partial \lambda_k^{(r_k-1)-j}} \Gamma(\lambda_k,\mu_k) \\ 0 & \Gamma(\lambda_k,\mu_k) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\partial \Gamma(\lambda_k,\mu_k)}{\partial \lambda_k} + \frac{\partial \Gamma(\lambda_k,\mu_k)}{\partial \mu_k} \\ 0 & \cdots & 0 & \Gamma(\lambda_k,\mu_k) \end{bmatrix}$$

where

$$\Gamma(J_{1_k}, J_{2_k}) = \int_0^\infty e^{-J_{2_k} t} t^{J_{1_k} - I} dt \tag{3.6}$$

Proof: Since the functions e^{at} and t^b $(a, b \in \mathbb{C})$ are defined on $\sigma(Q)$ and $\sigma(P)$ respectively, then by definition (2.1) we have,

$$e^{-J_{2_k} t} = \begin{bmatrix} e^{-\mu_k t} & \frac{d}{d\mu_k} e^{-\mu_k t} & \cdots & \frac{d^{r_k-1}}{d\mu^{r_k-1}} \\ 0 & e^{-\mu_k t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{d}{d\mu_k} e^{-\mu_k t} \\ 0 & \cdots & 0 & e^{-\mu_k t} \end{bmatrix}$$

and,

$$t^{J_{1_k}-I}=\begin{bmatrix} t^{\lambda_k-1} & \frac{d}{d\lambda}t^{\lambda_k-1} & \cdots & \frac{d^{r_k-1}}{d\lambda^{r_k-1}}t^{\lambda_k-1} \\ 0 & t^{\lambda_k-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{d}{d\lambda}t^{\lambda_k-1} \\ 0 & \cdots & 0 & t^{\lambda_k-1} \end{bmatrix}$$

so,

$$e^{-J_{2_k}t}t^{J_{1_k}-I}=$$

$$\begin{bmatrix} e^{-\mu_k t} & \frac{d}{d\mu_k}e^{-\mu_k t} & \cdots & \frac{d^{r_k-1}}{d\mu^{r_k-1}} \\ 0 & e^{-\mu_k t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{d}{d\mu_k}e^{-\mu_k t} \\ 0 & \cdots & 0 & e^{-\mu_k t} \end{bmatrix}\begin{bmatrix} t^{\lambda_k-1} & \frac{d}{d\lambda}t^{\lambda_k-1} & \cdots & \frac{d^{r_k-1}}{d\lambda^{r_k-1}}t^{\lambda_k-1} \\ 0 & t^{\lambda_k-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{d}{d\lambda}t^{\lambda_k-1} \\ 0 & \cdots & 0 & t^{\lambda_k-1} \end{bmatrix}$$

$$=\begin{bmatrix} e^{-\mu_k t}t^{\lambda_k-1} & \frac{\partial e^{-\mu_k t}t^{\lambda_k-1}}{\partial\lambda_k}+\frac{\partial e^{-\mu_k t}t^{\lambda_k-1}}{\partial\mu_k} & \cdots & \sum_{j=0}^{r_k-1}\frac{\partial^j\partial^{(r_k-1)-j}}{\partial\mu_k^j\partial\lambda_k^{(r_k-1)-j}}e^{-\mu_k t}t^{\lambda_k-1} \\ 0 & e^{-\mu_k t}t^{\lambda_k-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{\partial e^{-\mu_k t}t^{\lambda_k-1}}{\partial\lambda_k}+\frac{\partial e^{-\mu_k t}t^{\lambda_k-1}}{\partial\mu_k} \\ 0 & \cdots & 0 & e^{-\mu_k t}t^{\lambda_k-1} \end{bmatrix}$$

The result follows directly by the integration from 0 to ∞ the matrix above□.

Theorem 3.3 Given positive stable matrices P and Q with eigenvalues λ and μ respectively, then when $Re\ \mu\ \rightarrow\infty$ and λ is fixed then

$$\frac{\partial^p\partial^q\Gamma(\lambda,\mu)}{\partial\mu^p\,\partial\lambda^q}\sim -q\mu^{-\lambda-1}\Gamma^{(q-1)}(\lambda) \tag{3.7}$$

Proof

$$\frac{\partial^p\partial^q\Gamma(\lambda,\mu)}{\partial\mu^p\,\partial\lambda^q}=\frac{\partial^p\partial^q}{\partial\,\mu^p\partial\,\lambda^q}\Gamma(\lambda)$$

$$=\frac{\partial^p}{\partial\,\mu^p}\sum_{l=0}^{q}\binom{q}{l}\mu^{-\lambda}(-log\mu)^l\Gamma^{(q-l)}(\lambda)$$

$$=\frac{\partial^p}{\partial\,\mu^p}\sum_{l=0}^{q}\binom{q}{l}\mu^{-\lambda}(-log\mu)^l\Gamma^{(q-l)}(\lambda) \tag{3.8}$$

$$=\sum_{r=0}^{p}\sum_{l=0}^{q}\binom{q}{l}\binom{p}{r}\frac{\partial^{p-r}\mu^{-\lambda}}{\partial\,\mu^{p-r}}\frac{d^r(-log\mu)^l}{d\mu^r}\Gamma^{(q-l)}(\lambda)$$

To complete the prove, we need to find the leading term of equation (3.9) which we get when $r = p$ and $l = 1$ is

$$-\binom{q}{l}\binom{p}{r}\mu^{-\lambda-1}\Gamma^{(q-l)}(\lambda)$$

Thus

$$\frac{\partial^p \partial^q \Gamma(\lambda,\mu)}{\partial\mu^p \partial\lambda^q} = -q\mu^{-\lambda-1}\Gamma^{(q-l)}(\lambda) +$$

$$\sum_{r=0}^{p}\sum_{l=0}^{q}\binom{q}{l}\binom{p}{r}\frac{\partial^{p-r}\mu^{-\lambda}}{\partial\mu^{p-r}}\frac{d^r(-log\mu)^l}{d\mu^r}\Gamma^{(q-l)}(\lambda) \qquad (3.9)$$

$$= -q\mu^{-\lambda-1}\Gamma^{(q-l)}(\lambda)$$

$$\left[1+\frac{1}{-q\mu^{-\lambda-1}\Gamma^{(q-l)}(\lambda)}\sum_{r=0}^{p-1}\sum_{l=2}^{q}\binom{q}{l}\binom{p}{r}\frac{\partial^{p-r}}{\partial\mu^{p-r}}\left[\frac{d^r}{d\mu^r}(-log\mu)^l\Gamma^{(q-l)}(\lambda)\right]\right. \qquad (3.10)$$

$$= (q)(-\mu^{-\lambda-1})\Gamma^{(q-l)}(\lambda)\left[1+o\left\{\frac{log\mu}{\mu}\right\}\right] \qquad (3.11)$$

when $R_e\mu \to \infty$ we have

$$\frac{\partial^p}{\partial\mu^p}\frac{\partial^q}{\partial\lambda^q}\mu^{-\lambda}\Gamma(\lambda) \sim (q)(-\mu^{-\lambda-1})\Gamma^{(q-l)}(\lambda)$$

Definition 3.2 Given P_1, P_2 to be two matrices in$\mathbb{C}^{r\times r}$ and $P_1P_2 = P_2P_1$, then the Beta function $\beta(P_1, P_2)$ of P_1, P_2 is defined as

$$\beta(P_1, P_2) = \int_0^1 (1-v)^{P_1-I}\, v^{P_2-I}dv \qquad (3.12)$$

The definition of $\beta(P_1, P_2)$ is well defined see L,Jodar and J. G. Cortes [13]. □

Now we are ready to obtain a more general case for beta matrix functions.

Theorem 3.4 :Let $\Gamma(P_1, Q)$ and $\Gamma(P_2, Q)$ be such that, P_1, P_2 and Q are three positive stable matrices and P_1, P_2 and Q commutes. Then

$$\Gamma(P_1, Q)\Gamma(P_2, Q) = \beta(P_1, P_2)\Gamma(P_1 + P_2, Q) \qquad (3.13)$$

Proof

$$\Gamma(P_1, Q)\Gamma(P_2, Q) = \int_0^\infty e^{-xQ}\, x^{P_1-I}dx \int_0^\infty e^{-yQ}\, y^{P_2-I}dy$$

$$= \int_0^\infty\int_0^\infty e^{-(x+y)Q}\, x^{P_1-I}y^{P_2-I}dydx$$

Let $x + y = u$ and $y = uv$. So we have

$$\Gamma(P_1,Q)\Gamma(P_2,Q)=\int_0^\infty\int_0^\infty e^{-uQ}\left[u(1-v)^{P_1-I}(uv)^{P_2-I}\right]ududv$$

$$=\int_0^1 (1-v)^{P_1-I}\,v^{P_2-I}dv\int_0^\infty e^{-uQ}\,u^{P_1+P_2-I}du$$

$$=\beta(P_1,P_2)\Gamma(P_1+P_2,Q)$$

Conclusion

Matrix functions have a major role in science and engineering. One of the fundamental matrix functions, which is particularly important due to its connections with certain matrix differential equations and other special matrix functions, is the matrix Gamma function. This research article we concluded that,

1. The generalized gamma matrix function via Jordan canonical form.
2. Asymptotic approximation for the gamma matrix function with two positive stable matrices.
3. A more general case of beta matrix functions with two positive stable matrices.

References

[1] Abul-Dahab, and Bakhet A. K. A Certain Generalized Gamma Matrix Functions and Their Properties 20 Sep. 2014, Revised: 13 Nov. 2014, Accepted: 10 Dec. 2014. Published online: 1 Jan. 2015

[2] Andrews G.E. , Askey R. and Roy R. Special Function, Cambridge Univ. Press, rolethe Hypergeometric Matrix Function,Advances In Linear Algbra And Matrix Theory, (2014) ,4,134-141.

[3] Batahan R. S. Generalized Form Of Hermite Matrix Polynomials Via the Hypergeometric Matrix Function,Advances In Linear Algbra And Matrix Theory, (2014) ,4,134-141

[4] Defez E. and J´odar, L. Some Applications of the Hermite Matrix Polynomials Series Expansions, Journal Of Computational Applied Mathematics 99, 105-117, (1998). 11

[5] Defez E. , and J´odar L. Chebyshev Matrix Polynomials And Second Order Matrix Differential Equation. Utilitas Mathematica, (2002). 61. 107-123.

[6] Dunford N., Schwartz J. Linear Operators, vol.1, Interscience, New York, (1957)

[7] Erd´elyi A. Asymptotic Expansions, Dover Publications, Inc, (1956).

[8] Golub G., Van Loan C. F. Matrix Computations., The Johns Hopkins Univ. Press, Baltimore, MA, (1989).

[9] Gupta A. K. and. Nagar D. K, Matrix Variate Distributions. Boca Raton: Chapman & Hall/CRC, 2000. 54, 56

[10] Hassairi A and Regaig O. , “Characterizations of the beta distribution on symmetric matrices,” J. Multivariate Anal., vol. 100, no. 8, pp. 1682–1690, 2009. 54

[11] Higham N. J. Functions of A matrix Theory And Computation, Siam , phiadelphia,(2008)

[12] J´odar L. and Company R. Hermite Matrix Polynomials And Second Order Matrix Differential Equation, Approximation Theory and its Application 12(2),20-30,(1996).

[13] J´odar L. and Cort´e J. G. , Some Properties Of Gamma And Beta Matrix Function, Appl. Math. Lett, 11, (1998), 89-93

[14] J´odar L. and Cort´es J. G. On the Hypergeometric Matrix Function, J. Comp. Appl. Math.99, (1998),205-217.

[15] J`odar L. and Sastre J. The growth Of Laguerre Matrix Polynomials On Bounded Intervals, appl. math. lett, 13, (2000),21-26.

[16] Nagar D. K. Gupta A. K. and Sánchez L. E. “A class of integral identities with Hermitian matrix argument,” Proc. Amer. Math. Soc., vol. 134, no. 11, pp. 3329–3341, 2006. 54

[17] Nagar D. K, Roldán Correa A. and Gupta A. K.“Extended matrix variate gamma and beta functions,” J. Multivariate Anal., vol. 122, pp. 53–69, 2013. 54, 67, 69, 70, 79

[18] Nagar D. K and Correa A. R.“Extended matrix variate beta distributions,” Progress in Applied Mathematics, vol. 6, no. 1, pp. 40–53, 2013. 54, 79

[19] Nagar D.K , Morán-Vásquez R.A. and. Gupta A.K, “Ex- tended matrix variate hypergeometric functions and matrix variate distri- butions,” Int. J. Math. Math. Sci., vol. 2015, Article ID 190723, 15 pages, 2015. 54

[20] Nagar D. K, Gómez-Noguera and Gupta A. K. Generalized Extended Matrix Variate Beta and Gamma Functions and Their Applicationsing. cienc., vol. 12, no. 24, pp. 51–82, julio-diciembre. 2016.

[21] Olver F. W. J. Asymptotics And Special Functions, Academic Press, New York, (1974).

[22] Rainville E.D. Special Functions,the Macmillan Company, New York, (1960).

[23] Wasow W. Asymptotic Expansions For Ordinary Differential Equations, John Wiley And Sons,Inc. (1965).