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Explicit Finite Difference Method for the Two-dimensional Time Fractional Diffusion-Wave Equation

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Explicit Finite Difference Method for the Two-dimensional Time Fractional Diffusion-Wave Equation

Abstract:

In this paper, the time fractional diffusion-wave equation (TFDWE) is numerically studied, where the fractional derivative is defined in the sense of the Caputo. An explicit finite difference method (EFDM) for TFDWE is presented. The stability and the error analysis of the EFDM are discussed. To demonstrate the effectiveness of the approximated method, the test example is presented.

Keywords: Two-dimensional fractional diffusion-wave equation; Explicit finite difference method; von Neumann stability analysis.

المخلص:

في هذا البحث تم إستخدام طريقة الفروق المنتهية الصريحة لإيجاد الحلول العددية للمعادلات التفاضلية الجزئية ذات الرتب الكسرية, حيث قمنا بوصف المشتقات الكسرية بإستخدام مفهوم كابتو(Caputo). إضافة إلى ذلك قدمنا نظريتين مع إثباتهما وإستخدامهما في دراسة إستقرار الطريقة المستخدمة وتحديد الخطأ الناتج, ويتضمن البحث توضيحا لمدى فعالية الطريقة المستخدمة على مثال عددي.

الكلمات المفتاحية:

المعادلة التفاضلية الجزئية ذات الرتب الكسرية في فراغ ثنائي البعد، طريقة الفروق المنتهية الصريحة (القياسية), فون نيومان لتحليل الإستقرار.

I. Introduction

It's well known that fractional derivatives in mathematics are natural extension of integer-order derivatives, where the order is non integer [9]. Fractional order differential equations have been the focus of many studies due to their frequent appearance in various applications especially in the fields of fluid mechanics, viscoelasticity, biology, physics and engineering, see ([1], [5], [7], [9], [10]) and the references sited thesis. When a fractional derivative of order $1 < \alpha < 2$ replaces the second derivative in a diffusion-wave model ([2], [3], [4], [14], [20]). Analytic closed-form solutions for these initial-boundary value problems are elusive. Difference methods and, in particular, explicit finite difference methods, are an important class of numerical methods for solving fractional differential equations ([6], [15], [16]). The usefulness of the explicit method and the reason why they are widely employed is based on their particularly attractive features ([17], [19]).

A number of studies on the fractional diffusion and diffusion-wave equations have been carried out, see ([8],[11],[12],[13],[18]).

In this paper, EFDM scheme is designed for solving a two-dimensional fractional order diffusion-wave equation where the fractional derivative is in the Caputo sense. Moreover, since the explicit methods may be unstable, then, it is crucial to determine under which conditions, if any, these methods are stable. We will use here a kind of fractional von Neumann stability analysis to derive the stability conditions.

Consider the following two-dimensional fractional diffusion-wave equation:

$$\frac{u^{\alpha}u(x,y,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} + q(x,y,t),(x,y) \in \Omega, 0 < t \le T, (1)$$

$$u(x,y,0) = \psi(x,y), u_t(x,y,0) = \phi(x,y), (x,y) \in \overline{\Omega} = \Omega \,\mathrm{Y}\, \partial\Omega, (2)$$

$$u(x,y,t) = \varphi(x,y,t), (x,y) \in \partial\Omega, 0 < t \le T, \qquad (3)$$

where $1<\alpha<2$ the domain $\Omega=(0,L_1)\times(0,L_2)$, and $\partial\Omega$ is the boundary of $\Omega, \varphi(x,y,t), \psi(x,y), \phi(x,y)$, and $q(x,y,t)$ are known smooth functions.

Definition I.1. let $\alpha\in R^+, -\infty<a<b<\infty$, the Caputo Fractional Derivatives (CFDs) of order α are defined on $y(x)\in C^m[a,b]$ by [9].

$$({}_a^cD_x^\alpha y)(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)}{(x-t)^{1-n+\alpha}}dt, \qquad x>a, \qquad (leftCFD) \qquad (4)$$

$$({}_x^cD_b^\alpha y)(x)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{y^{(n)}(t)}{(t-x)^{1-n+\alpha}}dt, \qquad x<b, \qquad (rightCFD) \qquad (5)$$

Where $n=[\alpha]+1, \alpha\notin N_0$.

Also (4) and (5) are called the left-side and the right-side fractional derivatives in the Caputo sense, respectively

II. EFDM for TFDWE

In this work, the spatial $\alpha-$order fractional derivative is discretize using the Caputo formula [9], for $m-1<\alpha<m, m=2$:

$${}_0^CD_t^\alpha u(x,y,t)=\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\xi)^{1-\alpha}\frac{\partial^2 u(x,y,\xi)}{\partial \xi^2}d\xi, (6)$$

where $\Gamma(.)$ is the gamma function. Define $t_k=k_\tau, k=0,1,2,..., N; x_i=i\Delta x, i=0,1,2,..., M_1;$ $y_j=j\Delta y, j=0,1,2,..., M_2;$ where $\tau=\frac{T}{N}, \Delta x=\frac{L_1}{M_1}, \Delta y=\frac{L_2}{M_2}$, are time and space steps respectively, where M_1, M_2 and N are some given integers. Let $u_{i,j}^k$ be the numerical approximation to $u(x_i, y_j, t_k)$ and $q_{i,j}^k = q(x_i, y_j, t_k)$. For any grid function $u = \{u_{i,j}^k \mid 0 \le i \le M_1, 0 \le j \le M_2, 0 \le k \le N\}$.

In the differential equation (1), using

$$\frac{\partial^2 u(x,y,\xi)}{\partial \xi^2} = \frac{\partial^2 u(x,y,t_s)}{\partial \xi^2} + O(\tau), t_{s-1} \le \xi \le t_{s+1},$$

and

$$\frac{\partial^2 u(x,y,t_s)}{\partial \xi^2} = \frac{u(x,y,t_{s+1}) - 2u(x,y,t_s) + u(x,y,t_{s-1})}{\tau^2} + O(\tau),$$

the time fractional derivative term can be approximated by the following scheme:

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{k} \int_{s\tau}^{(s+1)\tau} \frac{\partial^2 u(x_i,y_j,\xi)}{\partial \xi^2} \frac{d\xi}{(t_{k+1}-\xi)^{\alpha-1}}$$

$$\approx \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{k} \int_{s\tau}^{(s+1)\tau} \frac{\partial^2 u(x_i,y_j,t_s)}{\partial \xi^2} \frac{d\xi}{(t_{k+1}-\xi)^{\alpha-1}}$$

$$\approx \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{k} \frac{u(x_i,y_j,t_{s+1}) - 2u(x_i,y_j,t_s) + u(x_i,y_j,t_{s-1})}{\partial \xi^2} \int_{s\tau}^{(s+1)\tau} \frac{d\xi}{(t_{k+1}-\xi)^{\alpha-1}}$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{k} \frac{u(x_i,y_j,t_{s+1}) - 2u(x_i,y_j,t_s) + u(x_i,y_j,t_{s-1})}{\tau^2} \int_{(k-s)}^{(k-s+1)} \frac{d\eta}{\eta^{\alpha-1}}$$

$$= \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{k} \frac{u(x_i,y_j,t_{k+1-s}) - 2u(x_i,y_j,t_{k-s}) + u(x_i,y_j,t_{k-1-s})}{\tau^2} \int_{s\tau}^{(s+1)\tau} \frac{d\eta}{\eta^{\alpha-1}}$$

$$= \frac{1}{\Gamma(3-\alpha)} \sum_{s=0}^{k} \frac{u(x_i,y_j,t_{k+1-s}) - 2u(x_i,y_j,t_{k-s}) + u(x_i,y_j,t_{k-1-s})}{\tau^2} \left[(s+1)^{2-\alpha} - (s)^{2-\alpha}\right]$$

$$= \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)} \left[u(x_i,y_j,t_{k+1}) - 2u(x_i,y_j,t_k) + u(x_i,y_j,t_{k-1})\right]$$

$$+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{s=0}^{k}\left[u(x_i,y_j,t_{k+1-s})-2u(x_i,y_j,t_{k-s})+u(x_i,y_j,t_{k-1-s})\right]$$

where $b_s=(s+1)^{2-\alpha}-(s)^{2-\alpha}, s=0,1,2,\ldots,N.$

Now the discrete of (1) using the explicit finite difference scheme can be written as

$$\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(u_{i,j}^{k+1}-2u_{i,j}^{k}+u_{i,j}^{k-1})+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{s=1}^{k}(u_{i,j}^{k+1-s}-2u_{i,j}^{k-s}+u_{i,j}^{k-1-s})b_s$$

$$=\frac{1}{(\Delta x)^2}(u_{i+1,j}^{k}-2u_{i,j}^{k}+u_{i-1,j}^{k})+\frac{1}{(\Delta y)^2}(u_{i,j+1}^{k}-2u_{i,j}^{k}+u_{i,j-1}^{k})+q_{i,j}^{k}+T(x,y,t),(7)$$

where $T(x,y,t)$ is the truncation term [8],

$$u_{i,j}^{k+1}=2u_{i,j}^{k}-u_{i,j}^{k-1})-\sum_{s=1}^{k}b_s(u_{i,j}^{k+1-s}-2u_{i,j}^{k-s}+u_{i,j}^{k-1-s})$$

$$+\overline{s_1}(u_{i+1,j}^{k}-2u_{i,j}^{k}+u_{i-1,j}^{k})+\overline{s_2}(u_{i,j+1}^{k}-2u_{i,j}^{k}+u_{i,j-1}^{k})+\Gamma(3-\alpha)\tau^{\alpha}q_{i,j}^{k},(8)$$

Where $\overline{s_1}=\Gamma(3-\alpha)s_1, s_1=\frac{\tau^{\alpha}}{(\Delta x)^2};\quad \overline{s_2}=\Gamma(3-\alpha)s_2, s_2=\frac{\tau^{\alpha}}{(\Delta y)^2}.$

III. Stability Analysis of EFDM

In this section we use the von Neumann method to study the stability analysis of the explicit finite difference scheme (6).

Theorem 1. *The explicit finite-difference scheme (6) for TFDWE is conditionally stable if*

$$\tau^{\alpha}\le S,$$

where

$$S=\frac{(\Delta x)^2(\Delta y)^2[1+\sum_{s=1}^{k}b_s(-1)^{-s}]}{\Gamma(3-\alpha)[(\Delta y)^2\sin^2(\frac{q_1\Delta_x}{2})+(\Delta x)^2\sin^2(\frac{q_2\Delta y}{2})]}.$$

Proof. Let us analyze the stability of (6) by substituting in a separated solution

$$u_{i,j}^{k} = \zeta_k e^{mq_1 i\Delta x} e^{mq_2 j\Delta y} = \zeta_k e^{mq_1 i\Delta x + mq_2 j\Delta y}$$

where $m = \sqrt{-1}, q_1, q_2$

are real spatial wave-number.

Inserting this expression, we get

$$\zeta_{k+1} e^{mq_1 i\Delta x + mq_2 j\Delta y} = 2\zeta_k e^{mq_1 i\Delta x + mq_2 j\Delta y} - \zeta_{k-1} e^{mq_1 i\Delta x + mq_2 j\Delta y}$$

$$-\sum_{s=1}^{k} b_s (\zeta_{k+1-s} e^{mq_1 i\Delta x + mq_2 j\Delta y} - 2\zeta_{k-s} e^{mq_1 i\Delta x + mq_2 j\Delta y} + \zeta_{k-1-s} e^{mq_1 i\Delta x + mq_2 j\Delta y})$$

$$+\overline{s_1}(\zeta_k e^{mq_1 (i+1)\Delta x + mq_2 j\Delta y} - 2\zeta_k e^{mq_1 i\Delta x + mq_2 j\Delta y} - \zeta_k e^{mq_1 (i-1)\Delta x + mq_2 j\Delta y})$$

$$+\overline{s_2}(\zeta_k e^{mq_1 i\Delta x + mq_2 (j+1)\Delta y} - 2\zeta_k e^{mq_1 i\Delta x + mq_2 j\Delta y} - \zeta_k e^{mq_1 i\Delta x + mq_2 (j-1)\Delta y}), (9)$$

divided (7) by $e^{mq_1 i\Delta x + mq_2 j\Delta y}$

then we get:

$$\zeta_{k+1} = 2\zeta_k - \zeta_{k-1} - \sum_{s=1}^{k} b_s (\zeta_{k+1-s} - 2\zeta_{k-s} + \zeta_{k-1-s})$$

Using the known Euler's formula $e^{m\theta} = \cos\theta + m\sin\theta, m = \sqrt{-1},$ we have:

$$+\overline{s_1}(\zeta_k e^{mq_1\Delta x} - 2\zeta_k + \zeta_k e^{-mq_1\Delta x}) + \overline{s_2}(\zeta_k e^{mq_2\Delta y} - 2\zeta_k + \zeta_k e^{-mq_2\Delta y}).(10)$$

$$\zeta_{k+1} = 2\zeta_k - \zeta_{k-1} - \sum_{s=1}^{k} b_s(\zeta_{k+1-s} - 2\zeta_{k-s} + \zeta_{k-1-s})$$

$$+\overline{s_1}[\zeta_k(\cos(q_1\Delta x) + m\sin(q_1\Delta x)) - 2\zeta_k + \zeta_k(\cos(q_1\Delta x) - m\sin(q_1\Delta x))]$$

$$+\overline{s_2}[\zeta_k(\cos(q_2\Delta y) + m\sin(q_2\Delta y)) - 2\zeta_k + \zeta_k(\cos(q_2\Delta y) - m\sin(q_2\Delta y))].(11)$$

Under some simplifications, we can write the above equation in the following form:

$$\zeta_{k+1} = 2\zeta_k - \zeta_{k-1} - \sum_{s=1}^{k} b_s(\zeta_{k+1-s} - 2\zeta_{k-s} + \zeta_{k-1-s})$$

$$+\overline{s_1}[-4\sin^2(\frac{q_1\Delta x}{2})]\zeta_k + \overline{s_2}[-4\sin^2(\frac{q_2\Delta y}{2})]\zeta_k. \qquad (12)$$

In the von Neumann method, the stability analysis is carried out using the amplification factor η defined by:

$$\zeta_{k+1} = \eta\zeta_{k.} \qquad (13)$$

Sure, η depends on k. But, let us assume for the moment that, as in [17], η is independent of time. Then, inserting this expression into Eq. (12) one gets:

$$\eta\zeta_k = 2\zeta_k - \eta^{-1}\zeta_k - \sum_{s=1}^{k} b_s(\eta^{1-s}\zeta_k - 2\eta^{-s} + \eta^{-1-s}\zeta_k)$$

$$+\overline{s_1}[-4\sin^2(\frac{q_1\Delta x}{2})]\zeta_k + \overline{s_2}[-4\sin^2(\frac{q_2\Delta y}{2})]\zeta_k. \qquad (14)$$

divided by ζ_k to obtain the following formula of η :

$$\eta = 2 - \eta^{-1} - \sum_{s=1}^{k} b_s(\eta^{1-s} - 2\eta^{-s} + \eta^{-1-s}) - \overline{s_1}[4\sin^2(\frac{q_1\Delta x}{2})] - \overline{s_2}[4\sin^2(\frac{q_2\Delta y}{2})].(15)$$

The mode will be stable as long as $|\eta| \le 1,$ i.e.,

$$-1 \le 2-\eta^{-1}-\sum_{s=1}^{k} b_s(\eta^{1-s}-2\eta^{-s}+\eta^{-1-s})-4\overline{s_1}\sin^2(\frac{q_1\Delta x}{2})-4\overline{s_2}\sin^2(\frac{q_2\Delta y}{2}) \le 1, (16)$$

considering the time-independent limit value $\eta = 1,$ then:

$$-1 \le 2-(-1)^{-1}-\sum_{s=1}^{k} b_s((-1)^{1-s}-2(-1)^{-s}+(-1)^{-1-s})-4\overline{s_1}\sin^2(\frac{q_1\Delta x}{2})-4\overline{s_2}\sin^2(\frac{q_2\Delta y}{2}) \le 1, (17)$$

$$-1 \le 3-\sum_{s=1}^{k} b_s(-4(-1)^{-s})-4\overline{s_1}\sin^2(\frac{q_1\Delta x}{2})-4\overline{s_2}\sin^2(\frac{q_2\Delta y}{2}) \le 1, \quad (18)$$

$$-1 \le 3+4\sum_{s=1}^{k} b_s(-1)^{-s}-4\frac{\Gamma(3-\alpha)\tau^{\alpha}}{(\Delta x)^2}\sin^2(\frac{q_1\Delta x}{2})-4\frac{\Gamma(3-\alpha)\tau^{\alpha}}{(\Delta y)^2}\sin^2(\frac{q_2\Delta y}{2}) \le 1, (19)$$

$$1 \ge -1-2\sum_{s=1}^{k} b_s(-1)^{-s}+\frac{2\Gamma(3-\alpha)\tau^{\alpha}}{(\Delta x)^2}\sin^2(\frac{q_1\Delta x}{2})+\frac{2\Gamma(3-\alpha)\tau^{\alpha}}{(\Delta y)^2}\sin^2(\frac{q_2\Delta y}{2}) \ge 0, (20)$$

$$\tau^{\alpha} \le S, \quad (21)$$

Where

$$S = \frac{(\Delta x)^2(\Delta y)^2[1+\sum_{s=1}^{k} b_s(-1)^{-s}]}{\Gamma(3-\alpha)[(\Delta y)^2\sin^2(\frac{q_1\Delta x}{2})+(\Delta x)^2\sin^2(\frac{q_2\Delta y}{2})]}. \quad (22)$$

And

$$\Gamma(3-\alpha[(\Delta y)^2\sin^2(\frac{q_1\Delta x}{2})+(\Delta x)^2\sin^2(\frac{q_2\Delta y}{2})]>0. \quad (23)$$

Theorem 2. *The truncation error of TFDWE is:*

$$T(x,y,t)=O(\Delta t)+O(\Delta x)^2+O(\Delta y)^2.$$

Proof. From the definition of truncating error given by Eq. (7), whey have:

$$T(x,y,t)=\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}(u_{i,j}^{k+1}-2u_{i,j}^{k}+u_{i,j}^{k-1})+\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{s=1}^{k}(u_{i,j}^{k+1-s}-2u_{i,j}^{k-s}+u_{i,j}^{k-1-s})b_s$$

$$-\frac{1}{(\Delta x)^2}(u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k) - \frac{1}{(\Delta y)^2}(u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k). \qquad (24)$$

Evaluating (1) at the point (x_i, y_j, t_k), gives

$$[\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - q]_{(x_i, y_j, t_k)} = 0, \qquad (25)$$

by the difference equation

$$\Delta_t u_{i,j}^{k+1} - \Delta_x^2 u_{i,j}^k - \Delta_y^2 u_{i,j}^k - q_{i,j}^k = T(x_i, y_j, t_k). \qquad (26)$$

Neglecting the truncation error term $T(x_i, y_j, t_k)$, we get the explicit difference scheme (8). From (1-8) and (26), we get

$$[\frac{\partial^\alpha u}{\partial t^\alpha}|_{(x_i,y_j,t_k)} - \Delta_t u_{i,j}^{k+1}] - [\frac{\partial^2 u}{\partial x^2}|_{(x_i,y_j,t_k)} - \Delta_x^2 u_{i,j}^k] - [\frac{\partial^2 u}{\partial y^2}|_{(x_i,y_j,t_k)} - \Delta_y^2 u_{i,j}^k] = T(x_i, y_j, t_k), \qquad (27)$$

$$\Delta_t u_{i,j}^{k+1} = \frac{\partial^\alpha u}{\partial t^\alpha}|_{(x_i,y_j,t_k)} + O(\Delta t)^2, \qquad (28)$$

$$\frac{\partial^\alpha u}{\partial t^\alpha}|_{(x_i,y_j,t_{k+1})} = \frac{\partial^\alpha u}{\partial t^\alpha}|_{(x_i,y_j,t_k)} + \Delta_t \frac{d\partial^\alpha u}{dt \partial t^\alpha}|_{(x_i,y_j,t_k)} + O(\Delta t)^2, \qquad (29)$$

so that

$$\Delta_t u_{i,j}^{k+1} = \Delta_t u_{i,j}^k + O(\Delta t) + O(\Delta t)^2, \qquad (30)$$

$$\frac{\partial^2}{\partial x^2} u(x_i, y_j, t_k) = \Delta_x^2 u(x_i, y_j, t_k) + O(\Delta x)^2, \qquad (31)$$

$$\frac{\partial^2}{\partial y^2} u(x_i, y_j, t_k) = \Delta_y^2 u(x_i, y_j, t_k) + O(\Delta y)^2. \qquad (32)$$

We finally get from equations. (24)-(31) and (32) the following result

$$T(x, y, t) = O(\Delta t) + O(\Delta x)^2 + O(\Delta y)^2. \qquad (33)$$

□

IV. Numerical Result

Example 1: Consider the time fractional diffusion-wave equation:

$$\frac{u^{\alpha}u(x,y,t)}{\partial t^{\alpha}}=\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}+\sin(x)\sin(y)[\frac{\Gamma(\alpha+3)}{2}t^2+2t^{\alpha+2}],$$

$$(x,y)\in\Omega=(0,\pi)\times(0,\pi),\qquad 0<t\leq 1,$$

with the initial conditions

$$u(x,y,0)=\frac{\partial u(x,y,0)}{\partial t}=0,\qquad (x,y)\in\overline{\Omega},$$

$$u_t(x,y,t)=0,\qquad (x,y)\in\overline{\Omega},\qquad 0<t\leq 1.$$

The exact solution to this two-dimensional fractional diffusion-wave equation is given by [14],[20]:

$$u(x,y,t)=\sin(x)\sin(y)t^{\alpha+2}$$

Table 1: The maximum errors at $T_{end}=1$ and $\Delta x=\Delta y=\frac{\pi}{40}$ for Example 1

α	τ	maximum error
1.5	1/5	1.7333715E-1
	1/10	5.2182597E-2
	1/20	2.8703252E-2
	1/40	3.4199144E-2
	1/80	3.9507111E-2
1.75	1/5	1.3455679E-1
	1/10	4.0264669E-2
	1/20	1.4434756E-2
	1/40	4.3443323E-2
	1/80	5.5591053E-2

Table 2: The maximum errors at $\alpha = 1.1$ and CPU time of Example 1

τ	$\Delta x = \Delta y$	maximum error	CPU time(s)
$\frac{1}{1000}$	$\pi/4$	1.6232019E-2	80.915
	$\pi/8$	4.3015254E-3	370.356
	$\pi/16$	1.2987275E-3	1662.690
	$\pi/32$	5.4679655E-4	1703.379
	$\pi/64$	3.5873774E-4	46080.077

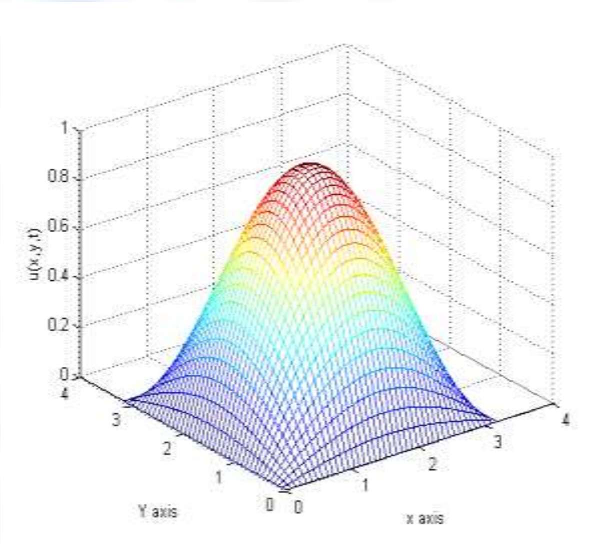

Figure 1: EFDM solution

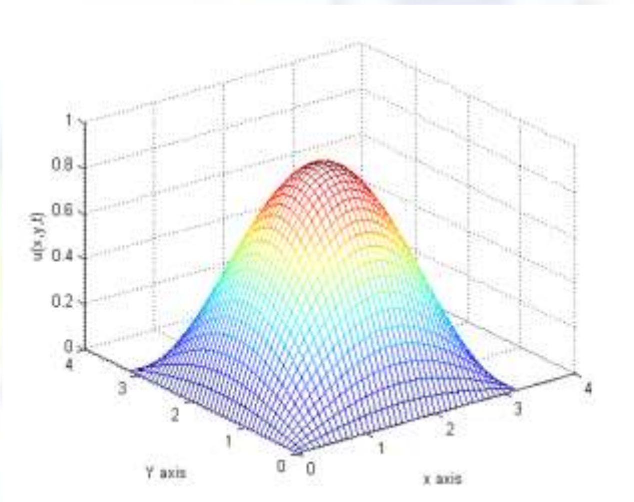

Figure 2: Exact solution

when $\Delta x = \Delta y = 0.08, \tau^{\alpha} = 0.001$ and $S = 0.65$.

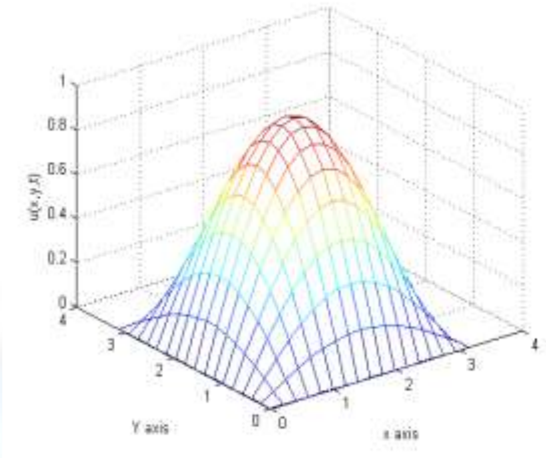

Figure 3: EFDM solution

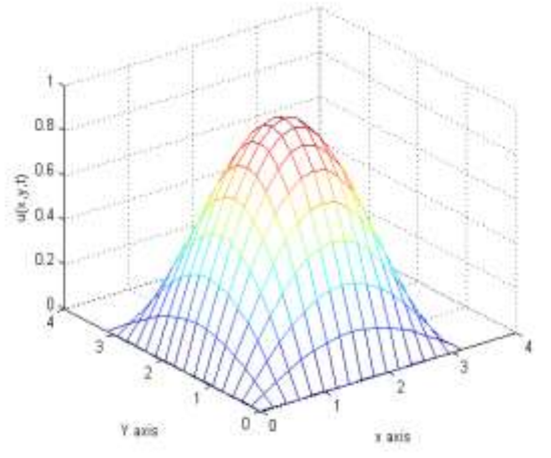

Figure 4: Exact solution

when $\Delta x = \Delta y = 0.20, \tau^{\alpha} = 0.0005$ and $S = 0.63.$

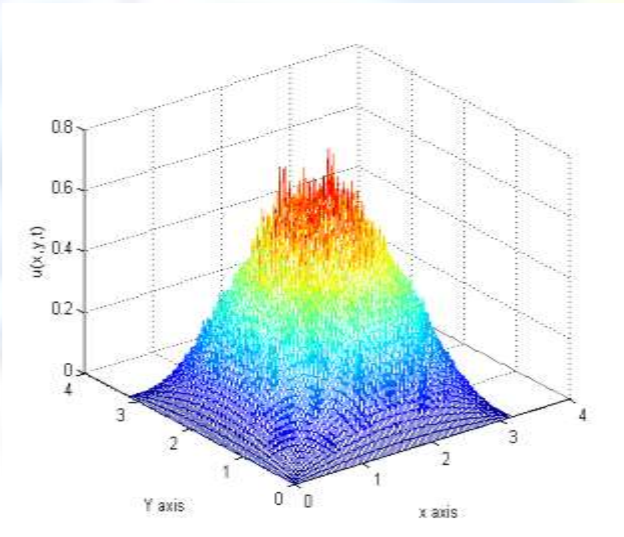

Figure 5: EFDM solution

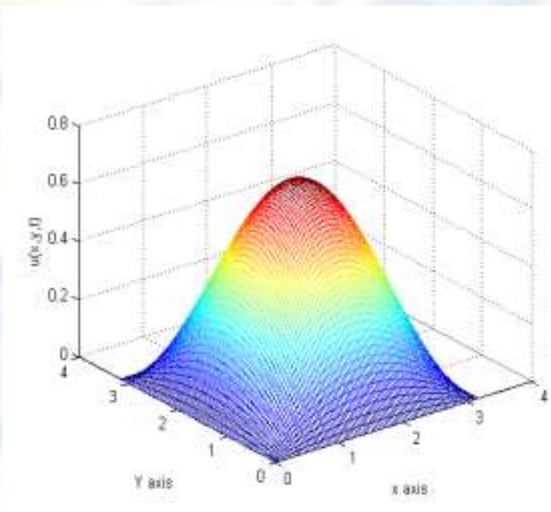

Figure 6: Exact solution

when $\Delta x = \Delta y = 0.013, \tau^{\alpha} = 0.079$ and $S = 0.062.$

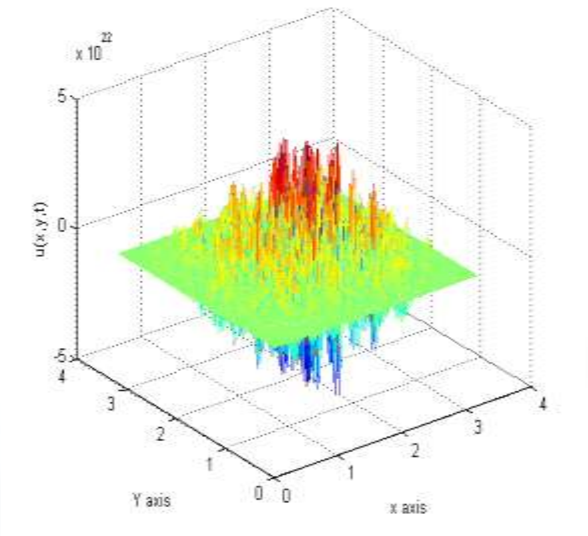

Figure 7: EFDM solution

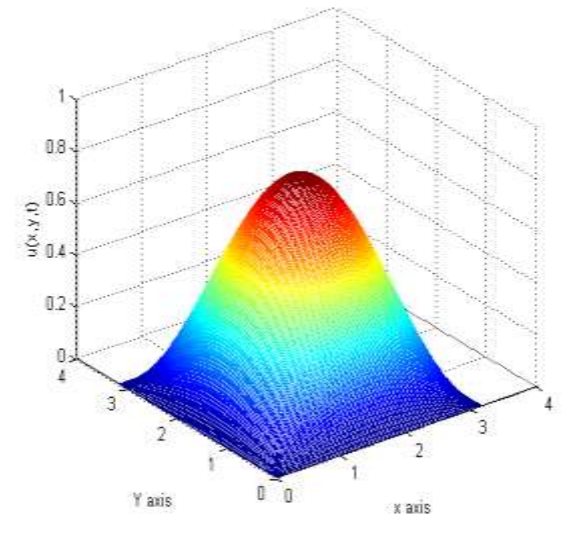

Figure 8: Exact solution

when $\Delta x = \Delta y = 0.007, \tau^{\alpha} = 0.02$ and $S = 0.001$.

The numerical studies are given as follows: Table 1 and Table 2 shows the maximum absolute numerical error, at time $T_{end} = 1$, between the exact solution and the numerical solution of the EFDM. In order to test the numerical scheme, Figure 1 shows the approximate solution where $\alpha = 1.5$, at $T_{end} = 1$, $\Delta x = \Delta y = 0.08$, $\tau^{\alpha} = 0.001$, $S = 0.65$, while Figure 2 shows the exact solution in this case. Figure 3 shows the approximate solution where $\alpha = 1.1$, at $T_{end} = 1$, $\tau^{\alpha} = 0.0005$, $\Delta x = \Delta y = 0.20$, and $S = 0.63$, while Figure 4 shows the exact solution in this case. Figure 5 shows the unstable solution behaviour when $\Delta x = \Delta y = 0.013$, $S = 0.62$, and $\tau^{\alpha} = 0.079$, where the value of is larger than the τ^{α} stability bound S, while Figure 6 shows the exact solution in this case, for more details on the stability conditions see Theorem 1. Figure 7 shows the unstable solution behaviour when e this larger than τ^{α} where the value of $S = 0.001$, and $\tau^{\alpha} = 0.02$, $\Delta x = \Delta y = 0.007$, stability bound S, while Figure 8 shows the exact solution in this case, for more details on the stability conditions see Theorem 1.

V. Conclusions

In this paper two-dimensional time fractional order diffusion-wave equation is studied using EFDM, where the fractional derivative is defined in the Caputo sense. Error analysis and stability of the explicit numerical method for TFDWE were discussed by means of a fractional version of the von Neumann stability analysis. The numerical result example is presented. These numerical result demonstrate that the EFDM is a computationally simple and effcient method for TFDWE.

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