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Solving the unsteady linear advection diffusion equations by using the totally volume integral of the local discontinuous Galerkin method.

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Highlights

- Integrals of the discretized equations have been unified by using the divergence theorem.
- The approach of reconstructing the solution on the boundaries of the cell has a major impact on the accuracy of the numerical solution.
- The proposed method reveals a good agreement with the exact solutions and previous studies
- Errors are decreasing as the mesh spacing is decreased as well as the order of the polynomial is increased.

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ABSTRACT

In this paper, we present the totally volume integral of the local discontinuous Galerkin TV-LDG method to solve the time-dependent linear convection-diffusion equation, the considered equation is discretized in space by the local discontinuous Galerkin method after the boundaries integral is transformed into the volume integral by employing the divergence theorem. The time discretization is accomplished by the third-order strong stability preserving Runge Kutta explicit SSP-RK (3, 3) method. Numerical solutions are compared with analytical solutions and other methods. The obtained results show that the totally volume integral of the local discontinuous Galerkin method is one of the most efficient methods for solving the time-dependent linear advection-diffusion equations.

1. Introduction

Fluid flow, heat, and mass transfer play a vital role in the natural environment and the engineering equipment. The pollution of the natural environment is mainly caused by mass and heat transfer, they also have a big impact on weather change like storms, floods, and fire when the fluid flow and heat transfer have an essential role. Predicting these processes help us in forecasting, and even controlling, potential dangers such as floods, tides, and fires. In all these cases, prediction offer economic benefits and contribute to human well-being. One of these prediction methods is an experimental investigation. Furthermore, it becomes possible to simulate very small or large-scale conditions such as fluid flow over planes, cars, ships, and the turbulent problems by employing the benefits and capabilities of computers to solve such complex physical problems which modeled by a set of partial differential equations PDEs taken as mathematical models for these physical systems. Here it is concerned with physical systems for which it is assumed that the basic equations describing their behavior are known theoretically, but for which no analytical solutions exist, and consequently, an approximate numerical solution will be sought instead. Many algorithms used for numerical simulation of physical problems solve discrete approximations of partial differential equations PDEs, the time-dependent linear convection-diffusion equation is one of these PDEs which is describing many physical situations.

Consider two-dimensional unsteady linear convection-diffusion partial differential equation

$$\frac{\partial u}{\partial t} + \beta_x \frac{\partial u}{\partial x} + \beta_y \frac{\partial u}{\partial y} = \alpha_x \frac{\partial^2 u}{\partial x^2} + \alpha_y \frac{\partial^2 u}{\partial y^2} \text{ on interval } \Omega = \{(x, y) | a \leq (x, y) \leq b\}, t > 0 \quad (1)$$

where $u(x, y, t)$ is a transported (advected and diffused) variable, β_x and β_y are the constant velocities of the solution in the x and y directions, α_x and α_y are the diffusion coefficients. Eq. (1) equation may be seen in computational fluid dynamics to model convection-diffusion of quantities such as mass, heat, energy, vorticity, the spread of contaminants in fluids, chemical separation processes and problems of environmental pollution. Many popular numerical methods introduced to approximate the considered PDE among them the finite difference methods introduced by Noye and Tan (1989) to solve the two-dimensional advection-diffusion equation by proposed a nine-point high-order compact implicit scheme for Eq. (1) which is third-order accurate in space and second-order accurate in time, and has a large zone of stability. Kalita et al. (2002) introduced a method to approximate Eq. (1) which is based on a high-order compact scheme and weighted time discretization. Their scheme is second or lowers order accurate in time and fourth-order accurate in space. The Local Discontinuous Galerkin LDG method has been introduced (Cockburn and Shu, 1998) to solve convection-diffusion problems involving second derivative viscous terms.

In this paper, we illustrate the essential ideas of the LDG method and how we can transform high order partial differential equations into a system of first-order partial differential equations by introducing a new auxiliary variable q to approximate the derivative of the solution u .

The main objective of this study is to develop the LDG by using the divergence theorem to unify the integrals of the governing equation (boundary integral and volume integral). The unified integrals and the local solvability of all the auxiliary variables are why the present method is called the totally volume integral local discontinuous Galerkin TV-LDG method. Since the first-order system of equations will discretize by using the TV-LDG space discretization method. Then the obtained system of ordinary differential equations will be integrated in time by using the strong stability preserving Runge-Kutta SSP-RK third-order time discretization method (Shu and Osher, 1988).

2. The totally volume integral of the local discontinuous Galerkin method

Consider the two-dimensional time-dependent linear convection-diffusion equation.

$$\frac{\partial u}{\partial t} + \frac{\partial f_x(u)}{\partial x} + \frac{\partial f_y(u)}{\partial y} = \alpha \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \quad (2)$$

where $f_x(u) = \beta_x u$ and $f_y(u) = \beta_y u$ are the cartesian components of the convective flux $f(u)$ and by introducing a new auxiliary variable $q = \alpha(\nabla u)$ we can rewrite Eq. (2) as a system of first-order equations.

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) - \nabla \cdot q = 0 \quad (3)$$

$$q - \alpha(\nabla u) = 0 \quad (4)$$

assuming we are solving these systems of Eq. (3) and Eq. (4) on interval $\Omega \in [a, b]$. We divide the domain $[a, b]$ into equally space N elements. First, partition the whole domain Ω into small computational cells $\Omega = \cup_{j=1}^N \Omega_j$, where Ω_j is the subdomain also called cell or element, the length of the cell for the one-dimensional domain is $h = \Delta x = [x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}]$, in case of two dimensional the mesh size is $h = \Delta x = \Delta y$. To seek the numerical solution for unknown quantity u which could be velocity or temperature. First of all, the spatial discretization is considered. The weak forms of the system of Eq. (3) and Eq. (4) are obtained by the scalar multiplication of the partial differential equations with test functions w and p then the integration by parts is applied over the subdomain Ω_j .

$$\int_{\Omega_j} w \frac{\partial u_h}{\partial t} d\Omega_j - \int_{\Omega_j} \nabla w (f(u_h) - q_h) d\Omega_j + \oint_{\partial\Omega_j} w (\hat{f} - \hat{q}) \cdot n d\partial\Omega_j = 0 \quad (5)$$

$$\int_{\Omega_j} p q_h d\Omega_j + \alpha \int_{\Omega_j} \nabla p u_h d\Omega_j - \alpha \oint_{\partial\Omega_j} (p \hat{u}) \cdot n d\partial\Omega_j = 0 \quad (6)$$

where $\partial\Omega_j$ the boundary of the element and n is the unit outward normal vector to the boundary and, all the “hat” terms \hat{f} , \hat{q} and \hat{u} are the numerical fluxes that designed to approximate the convective and diffusion fluxes at the boundaries of the element $\partial\Omega_j$.

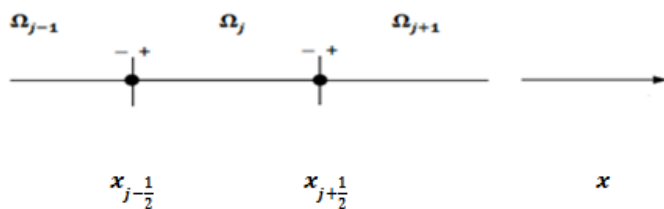


Fig. 1. Illustration of boundary interfacial quantities for element Ω_j

2.1. The convective numerical flux

The convective numerical flux \hat{f} is chosen to be the upwinding scheme which is dependent on the sign of the wave propagation speed and the numerical flux is defined as follows (Torro, 1999).

$$\hat{f}(u_h^-, u_h^+) = \begin{cases} f(u_h^-) & \text{if } \beta > 0 \\ f(u_h^+) & \text{if } \beta < 0 \end{cases} \quad (7)$$

2.2. The diffusion numerical flux

Now the diffusion numerical fluxes \hat{q} and \hat{u} are the approximations of q_h and u_h on the element boundaries and depend on the solution of both sides of the element interface $\hat{q} (q_{j+\frac{1}{2}}^+, q_{j+\frac{1}{2}}^-)$. As can be seen from Fig. 1 the plus sign means the numerical solution will be taken from the right-hand side and the negative sign is the numerical solution from the left-hand side. In this work, two different schemes of diffusion numerical flux are used to approximate the solution on the element boundaries.

2.2.1. Central fluxes scheme

Bassi and Rebay (1997) were the first to apply the discontinuous finite element method for the solution of diffusion-type problems. In their original approach, they proposed the simplest central flux expression, and applied the scheme for discontinuous flow calculations, obtaining quite satisfactory results. Their central scheme uses the average of the two values across the boundary

$$\hat{q}_{j+\frac{1}{2}} = \frac{1}{2} (q_h^+ + q_h^-)_{j+\frac{1}{2}} \quad (8)$$

$$\hat{u}_{j+\frac{1}{2}} = \frac{1}{2} (u_h^+ + u_h^-)_{j+\frac{1}{2}} \quad (9)$$

2.2.2. Alternating fluxes scheme

These selections may be interpreted as meaning that the available values at the neighboring boundary are considered known, and are applied as the boundary conditions for element Ω_j , whenever they become available during the iteration. This approach was first proposed by Cockburn and Shu (1998).

$\hat{q}_{j+\frac{1}{2}} = (q_h^+)_{j+\frac{1}{2}}$ (The numerical solution is taken from the right side of the node $(x_{j+\frac{1}{2}})$)

$\hat{u}_{j+\frac{1}{2}} = (u_h^-)_{j+\frac{1}{2}}$ (The numerical solution is taken from the left side of the node $(x_{j+\frac{1}{2}})$)

To unify the integrals (boundary integral and volume integral), the total volume integral of the numerical fluxes is used for this purpose. This can be done by using the relation between boundary and volume integrals for any vector \vec{F} , which is given by the divergence theorem.

$$\oint_{\partial\Omega} \vec{F} \cdot n d\partial\Omega = \iiint_{\Omega} \nabla \cdot \vec{F} d\Omega \quad (10)$$

by applying the divergence theorem to the boundary integrals of the numerical fluxes in Eq. (5) and Eq. (6). We obtain

$$\int_{\Omega_j} [w \frac{\partial u_h}{\partial t} + \nabla w (f(u_h) - q_h) + \nabla (w (\hat{f} - \hat{q}))] d\Omega_j = 0 \quad (11)$$

$$\int_{\Omega_j} [p q_h + (\nabla p) u_h - \nabla (p \hat{u})] d\Omega_j = 0 \quad (12)$$

the numerical solutions w, p, u_h, q_h and the physical flux $f(u_h)$ will be approximated as a combination of N_j basis functions in every element as:

$$U(x, y) = \sum_{i=1}^{N_j} \phi_i(x, y)(U_{h,i}) \tag{13}$$

by applying the numerical integration and assembling all the elemental contributions, the system of ordinary differential equations that govern the evolution in time of the discrete solution can be written as.

$$M \frac{du_h}{dt} = R(u_h) \tag{14}$$

where M is the mass matrix obtained after applying the numerical integration over the cell and u_h is the global vector of the degrees of freedom $R(u_h)$ is the residual of the process resulting from Eq. (11) and Eq. (12).

3. Time integration

The main idea of the totally volume Integral local discontinuous Galerkin TV-LDG method is that we can solve q explicitly and locally (in element Ω_j) in terms of u_h by inverting the element mass matrix inside the cell Ω_j . Thus, we eliminate the equation for q , and obtain the combined ordinary differential equation system for freedoms U_h as follows.

$$\frac{d}{dt} U_h = M^{-1}R(u_h) = L(U_h, t) \tag{15}$$

where this ordinary differential equation appears from the discretization of the spatial derivative in the partial differential equation. This semi-discretize scheme is discretized in time by using the third order strong stability preserving Runge-Kutta SSP-RK method (Shu and Osher, 1988), where U^n is the solution at the time t^n and the solution at the next time step is U^{n+1} which is obtained after the s stages. Where the time marching algorithm performs by using the three-stage third-order Runge–Kutta method as follows.

$$U^{(1)} = U^n + \Delta t \cdot L(U^n, t^n) \tag{16}$$

$$U^{(2)} = \frac{3}{4}U^n + \frac{1}{4}U^{(1)} + \frac{1}{4}\Delta t \cdot L(U^{(1)}, t^n + \Delta t) \tag{17}$$

$$U^{(3)} = \frac{1}{3}U^n + \frac{2}{3}U^{(2)} + \frac{2}{3}\Delta t \cdot L(U^{(2)}, t^n + \frac{1}{2}\Delta t) \tag{18}$$

4. Results and discussion

In this section, the solutions of time-dependent linear convection-diffusion equations will be investigated by the proposed method. Hence, to compare our numerical solutions with the analytical solutions we report the average and maximum errors;

$$L_1 = \frac{\sum_{j=1}^N \sum_{i=1}^{DOF} |U_{i,h} - U_{i,exact}|}{NDOF} \tag{19}$$

Table 1

The L_1 error and the order of accuracy for 1D linear advection-diffusion equation with periodic boundary conditions at time $t = 2$ by using polynomials of orders $k = 1$ to 3 with central flux.

Number of elements	Variable	$k=1$			$k=2$			$k=3$		
		L_1 error	ratio	order	L_1 error	ratio	order	L_1 error	ratio	order
10	u	5.9055E-03			1.2129E-04			1.0729E-05		
	q	1.1979E-02			2.4200E-04			2.7661E-05		
20	u	1.6998E-03	3.47	1.80	1.4768E-05	8.21	3.04	1.0244E-06	10.47	3.39
	q	3.7703E-03	3.18	1.67	3.1243E-05	7.75	2.95	2.0556E-06	13.46	3.75
40	u	4.6260E-04	3.67	1.88	1.8420E-06	8.02	3.00	8.6652E-08	11.82	3.56
	q	1.0363E-03	3.64	1.86	4.0178E-06	7.78	2.96	1.3583E-07	15.13	3.92
80	u	1.2188E-04	3.80	1.92	2.3089E-07	7.98	3.00	6.5823E-09	13.16	3.72
	q	2.6991E-04	3.84	1.94	5.1109E-07	7.86	2.97	8.6480E-09	15.71	3.97

$$L_\infty = \max |U_{i,h} - U_{i,exact}| \tag{20}$$

where $U_{i,exact}$ is the exact solution and $U_{i,h}$ is the numerical solution obtained by the TV-LDG method at every node in the domain, DOF is the degree of freedom over the element Ω_j , the number of elements is denoted as N , and $NDOF$ is the total number of nodes over the entire problem domain. The ratio of the errors is defined by.

$$Ratio = \frac{error(N_1)}{error(N_2)} \tag{21}$$

the order of convergence of the scheme is calculated by using the following formula.

$$Order = \frac{\log(error(N_1)/error(N_2))}{\log(N_2/N_1)} \tag{22}$$

where $error(N_1)$ and $error(N_2)$ are the errors for the numbers of cells N and $2 \times N$ respectively.

4.1. Test Example 1

The one-dimensional linear advection-diffusion equation with the periodic boundary condition, considered in Cockburn and Shu (1998).

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \tag{23}$$

with the initial condition as:

$$u(x, 0) = u_0(x) = \sin(x) \tag{24}$$

the exact solution is:

$$u(x, t) = \exp\{-\alpha t\} \sin(x - t) \tag{25}$$

We used the TV-LDG method to discretize the problem domain $[0, 2\pi]$ into N equally elements. The shape functions are constructed from the Lagrange polynomials of orders k from 1 to 3 and $\Delta t = cfl * h^2$. The SSP-RK (3, 3) is used for evaluating the time integral part. And we compute the solution up to a period ($t = 2$), where cfl is a parameter dependent on the case study and mesh size h , hence $cfl = 0.04$ for $k = 1$, $cfl = 0.02$ for $k = 2$ and $cfl = 0.01$ for $k = 3$.

A comparison of two different types of diffusion numerical fluxes is made (I). The central flux and, (II). The alternative flux in Table 1 and Table 2 to exhibit the average error L_1 and the order of accuracy. This gives the completed description of the error for u_h over the whole domain, as u_h in each element is a polynomial of degree k . Two main points could be taken away from Table 1 and, Table 2. First, it can be observed that the error is decreasing as the number of elements N is increased. Second, the approximate solution is converging to the exact solution with a proper rate.

Table 2

The L_1 error and the order of accuracy for 1D linear advection-diffusion equation with periodic boundary conditions at time $t = 2$ by using polynomials of orders $k = 1$ to 3 with alternative flux.

Number of elements	Variable	$k=1$			$k=2$			$k=3$		
		L_1 error	ratio	order	L_1 error	Ratio	order	L_1 error	ratio	order
10	u	2.8663E-03			1.5355E-04			5.6730E-06		
	q	6.3774E-03			1.3526E-04			5.0964E-06		
20	u	7.1098E-04	4.03	2.01	1.8733E-05	8.20	3.04	3.5066E-07	16.18	4.02
	q	1.5859E-03	4.02	2.01	1.7700E-05	7.64	2.93	3.3564E-07	15.18	3.92
40	u	1.7693E-04	4.02	2.01	2.3375E-06	8.01	3.00	2.1980E-08	15.95	4.00
	q	3.9585E-04	4.01	2.00	2.2624E-06	7.82	2.97	2.1480E-08	15.63	3.97
80	u	4.4288E-05	3.99	2.00	2.9095E-07	8.03	3.01	1.3727E-09	16.01	4.00
	q	9.9009E-05	4.00	2.00	2.8645E-07	7.90	2.98	1.3574E-09	15.82	3.98

Fig. 2 and Fig. 3 show the super-accurate method which states that the TV-LDG method converges to the order of accuracy equal to or higher than the number of the distribution points is proved to be accurate by all the polynomials that are studied in this test example. Moreover, in Fig. 4 the proposed scheme demonstrates a

good agreement with the analytical solution at different times by divided the domain into ten (linear $k=1$) elements and using alternative flux.

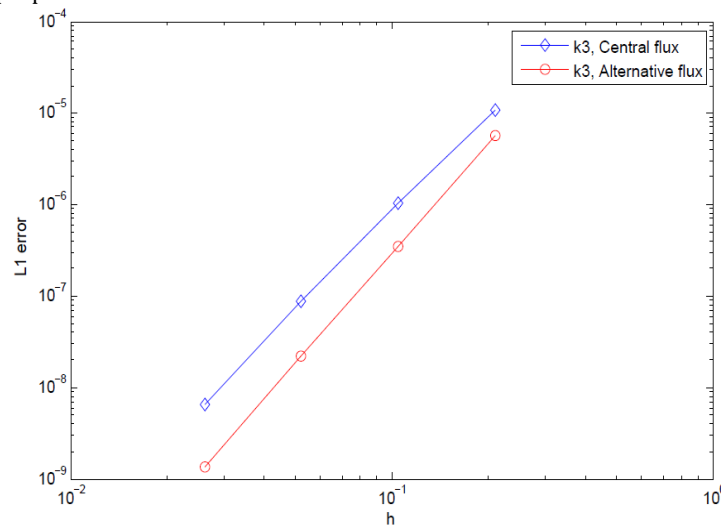


Fig. 2. The L_1 errors for polynomials of orders $k = 3$ at $t = 2$ for 1D linear advection-diffusion equation for case central and alternative fluxes to approximate the diffusion term.

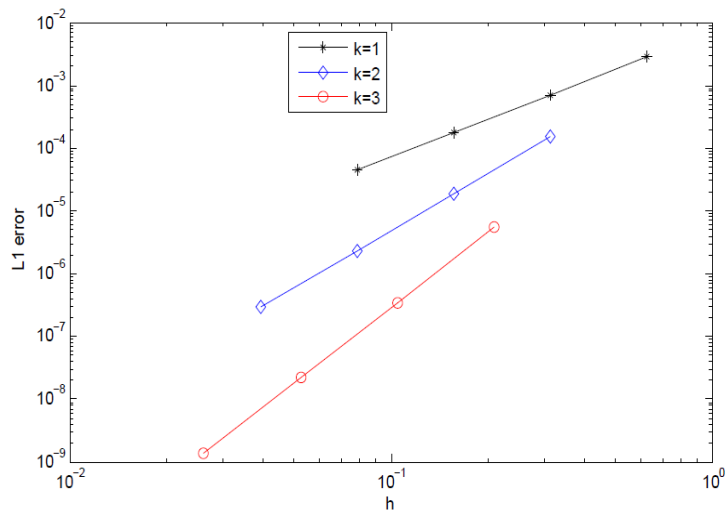


Fig. 3. The L_1 errors for polynomials of orders $k = 1$ to 3 at $t = 2$ for 1D linear advection-diffusion equation.

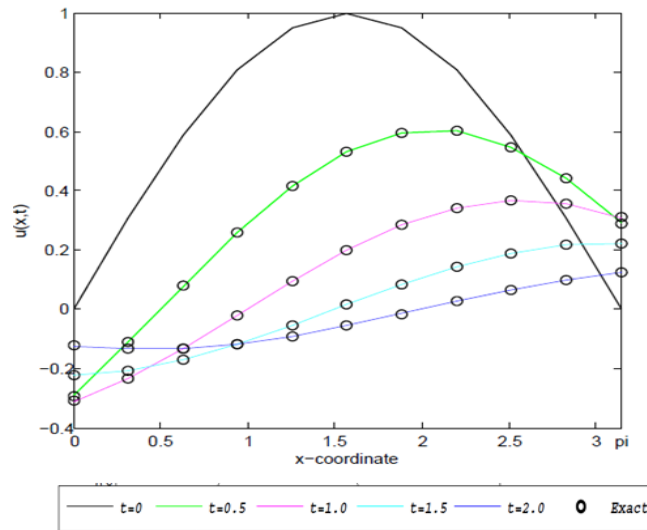


Fig.4. Comparison of exact with numerical results using $k = 1$ at $t = 2$ for 1D linear advection-diffusion equation.

4.2. Test example 2

The two-dimensional linear advection-diffusion equation with the following initial and boundary condition

$$u(x, y, 0) = \exp \left\{ -\frac{(x - 0.5)^2}{\alpha_x} - \frac{(y - 0.5)^2}{\alpha_y} \right\} \quad (26)$$

$$u(x, y, t) = \frac{1}{1 + 4t} \exp \left\{ -\frac{(x - \beta_x t - 0.5)^2}{\alpha_x} - \frac{(y - \beta_y t - 0.5)^2}{\alpha_y} \right\} \quad (27)$$

This problem has been considered in many references (Kalita et al., 2002; Karra and Zhang, 2004; Noye and Tan, 1988; Tian and Ge, 2007; Dehghan and Mohebbi, 2008). The exact and boundary conditions can be obtained from Eq. (27) in case of $\beta_x = \beta_y = 0.8$, $\alpha_x = \alpha_y = 0.01$. It can be seen that the initial condition is a Gaussian pulse as shown in Fig. 5. In Table 3 and Table 4 the errors and orders of accuracy are obtained by the TV-LDG method at $t = 0.5$ and $(x, y) \in [0, 1]$ by using (linear $k=1$) element and (quadratic $k=2$) element with different number of elements and by applying the central scheme for the diffusion numerical flux. It can be seen that the errors are decreasing as mesh spacing is decreased as well as the order of the polynomial is increased.

Table 3

The errors and the order of accuracy for problem 2 at time $t = 0.5$ by using $k=1$ polynomials.

Number of elements	Average error L_1	Order	Maximum error L_∞	Order
10×10	2.997E-03		6.278E-02	
20×20	5.603E-04	2.42	1.807E-02	1.80
40×40	1.323E-04	2.08	4.732E-03	1.93

Table 4

The errors and the order of accuracy for problem 2 at time $t=0.5$ by using $k=2$ polynomials.

Number of elements	Average error L_1	Order	Maximum error L_∞	Order
10×10	1.492E-04		2.802E-03	
20×20	1.698E-5	3.14	3.986E-04	2.81
40×40	2.333E-06	2.86	7.728E-05	2.37

Table 5 shows a comparison between the TV-LDG method and other numerical methods for the case of $\beta_x = \beta_y = 0.8$, and the diffusion coefficients $\alpha_x = \alpha_y = 0.01$ on the domain $\Omega = \{(x, y) | 0 \leq x, y \leq 2\}$, at final time $t = 1.25$. It can be noticed again the present scheme has a good agreement with the analytical solution and gives the lower error as compared with other methods.

Table 5

Errors obtained for Problem 2 at $t = 1.25$ and mesh size $\Delta x = \Delta y = h = 0.025$.

Method	Average error L_1	Maximum error L_∞
(Noye and Tan, 1989)	1.971E-05	6.509E-04
(Kalita et al., 2002)	1.597E-05	4.447E-04
(Tian and Ge, P-R ADI 2007)	3.109E-04	7.778E-03
(Karra and Zhang, 2004)	9.218E-06	2.500E-04
(Tian and Ge ADI, 2007)	9.663E-06	2.664E-04
(Dehghan and Mohebbi, 2008)	9.493E-06	2.476E-04
Present	2.5641E-06	6.9158E-05

Numerical solutions are obtained at different time $t=0.6$ and $t=1.25$. In this test example, we have taken $\beta_x = \beta_y = 0.8$ and $\alpha_x = \alpha_y = 0.01$ using the equally spaced elements with mesh size $h = \Delta x = \Delta y = 0.025$. Physical behaviors of the solution u at different time levels can be observed in Fig. 6 and Fig.7. As time increases, movement of wave is well captured by using the totally volume local discontinuous Galerkin method with linear element $k=1$

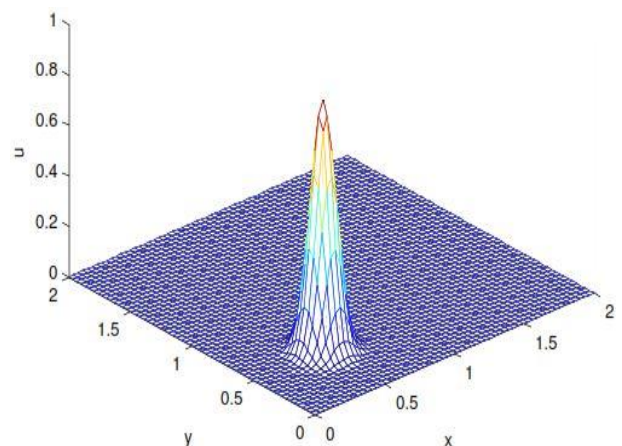


Fig. 5. The plot of the initial solution $u(x, y, 0)$ of example 2

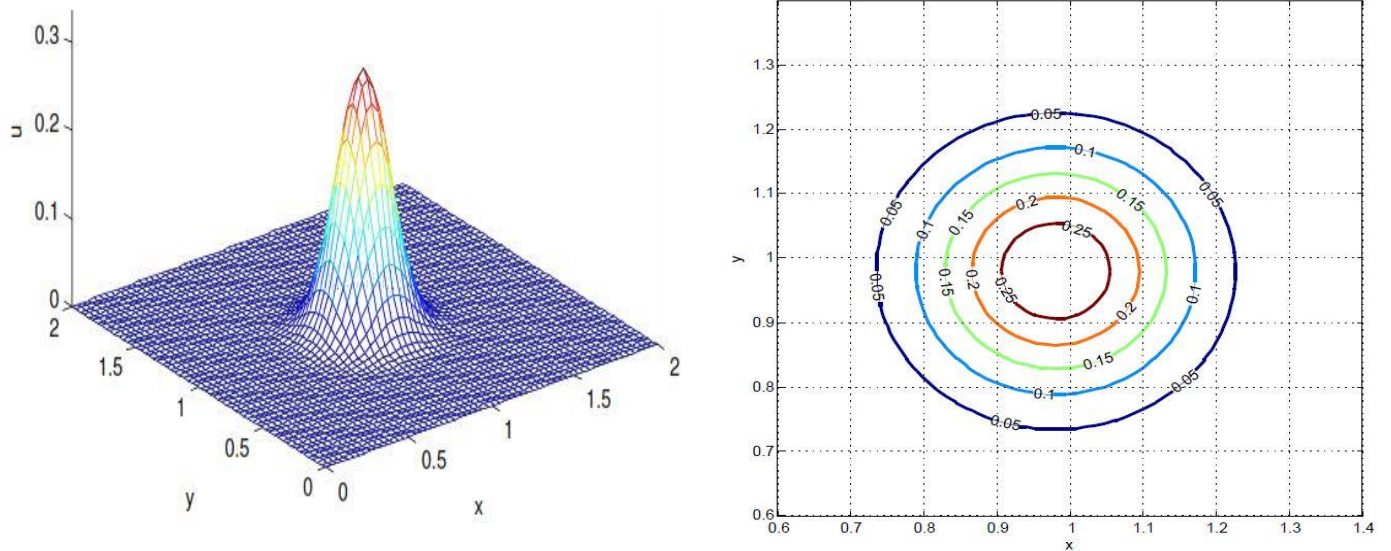


Fig. 6. The plot of example2 at $t=0.6$, with mesh size $\Delta x = \Delta y = \frac{1}{40}$ (left-hand side) the contour of the solution (right-hand side)

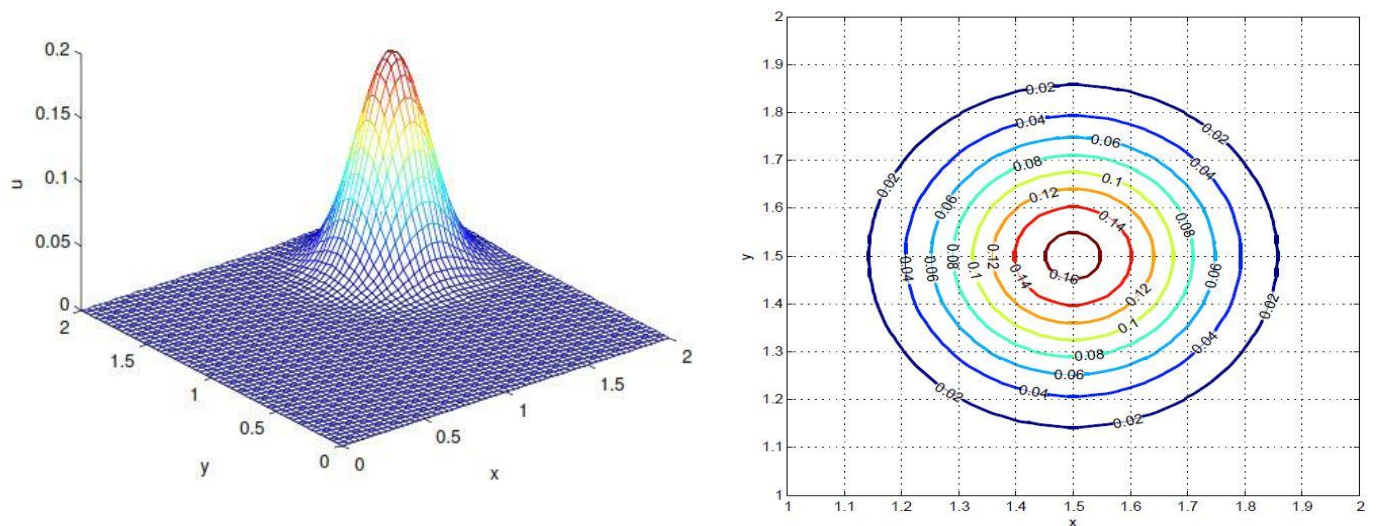


Fig. 7. The plot of example2 at $t=1.25$, with mesh size $\Delta x = \Delta y = \frac{1}{40}$ (left-hand side) the contour of the solution (right-hand side)

4.3. Test example 3

Consider the partial differential Eq. (1) with the following initial.

$$\begin{aligned}
 u(x, y) = & 5 \exp \left\{ -\frac{(9x-2)^2}{4} - \frac{(9y-2)^2}{4} \right\} \\
 & + 7 \exp \left\{ -\frac{(9x+1)^2}{50} - \frac{(9y+1)}{10} \right\} \\
 & + 4 \exp \left\{ -\frac{(9x-7)^2}{4} - \frac{(9y-3)^2}{4} \right\} \\
 & - 2 \exp \{ -(9x-4)^2 - (9y-7)^2 \}
 \end{aligned} \tag{28}$$

with a Dirichlet boundary type condition.

$$u(x, y, t) = u(x, y) - gt \tag{29}$$

where g is a positive constant. The above problem is encountered in many transport phenomena such as vorticity, advective, convective heat transfer and other diffusive-advective processes (Zerroukat and Djidjel, 2000).

To compare the solution with Zerroukat and Djidjel (2000) and Dehghan and Mohebbi (2008) the problem has been solved on $\Omega = \{(x, y) | 0 \leq (x, y) \leq 1\}$ where $g = 0.1$, $\beta_x = -0.1$, $\beta_y = 0.2$, $\alpha_x = 0.2$ and $\alpha_y = 0.3$. Where $u(x, y, t)$ is a scalar variable which is convected in the x and y directions with constant velocities β_x and β_y respectively and is spread with constant diffusivities α_x and α_y . In Fig. 8, the plot and contour plot of the initial distribution of $u(x, y, 0)$ have been shown.

Numerical solutions are obtained at different time $t=0.1$, $t=0.5$ and $t=1$ Fig. 9 to Fig. 11. In this test example, the central scheme has been applied to approximate the diffusion numerical flux and by using the equally spaced (linear $k=1$) elements with mesh size $h = \Delta x = \Delta y = 0.025$. Fig. 9 to Fig. 11 demonstrate that the obtained numerical solutions $u(x, y, t)$ are unconditionally stable and similar to the solutions of high-order compact boundary value method (Dehghan and Mohebbi, 2008) and implicit Crank-Nicolson thin-plate spline (Zerroukat and Djidjel, 2000).

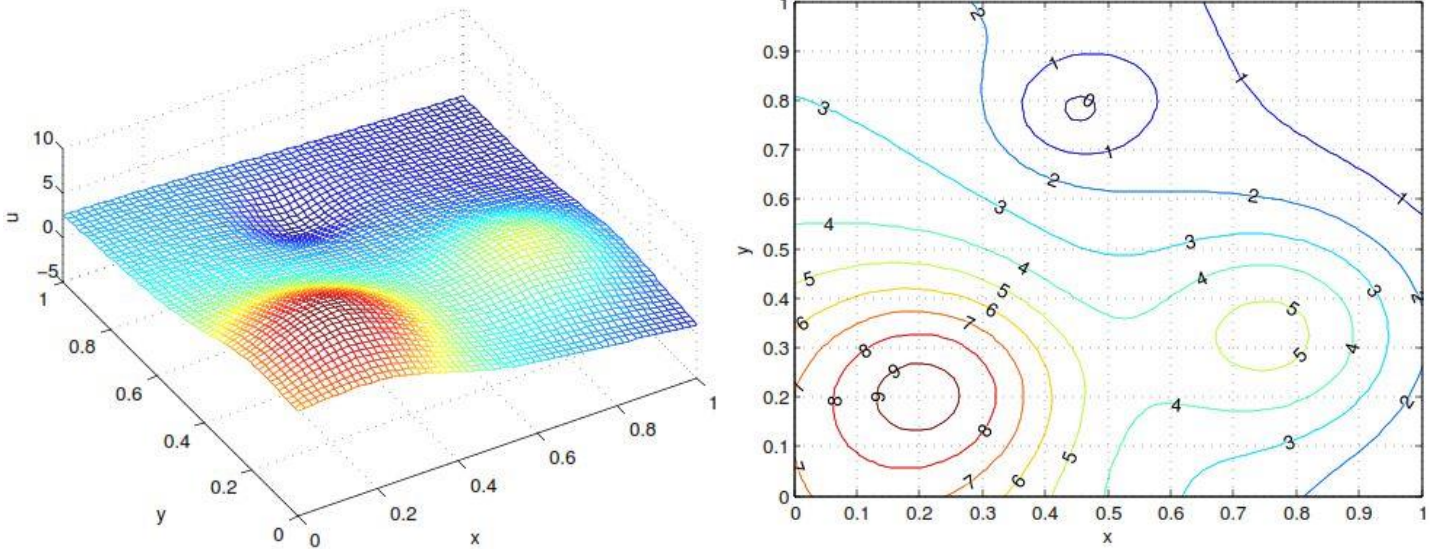


Fig. 8. A plot of the initial $u(x, y, 0)$ distribution (left-hand side). A contour plot of initial $u(x, y, 0)$ distribution (right-hand side) of problem 3.

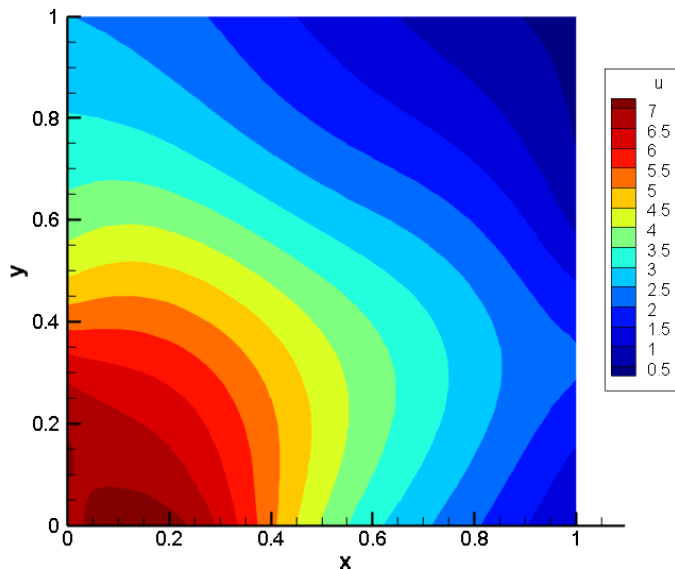


Fig. 9. The contour plot of approximate solution $u(x, y, t)$ with mesh size $\Delta x = \Delta y = 0.025$, $\Delta t = 0.0005$, $t = 0.1$

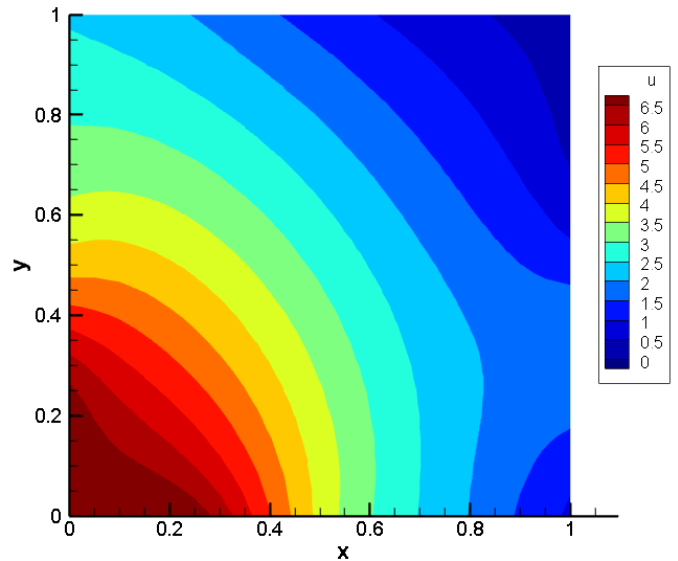


Fig. 11. The contour plot of approximate solution $u(x, y, t)$ with mesh size $\Delta x = \Delta y = 0.025$, $\Delta t = 0.0005$, $t = 1$

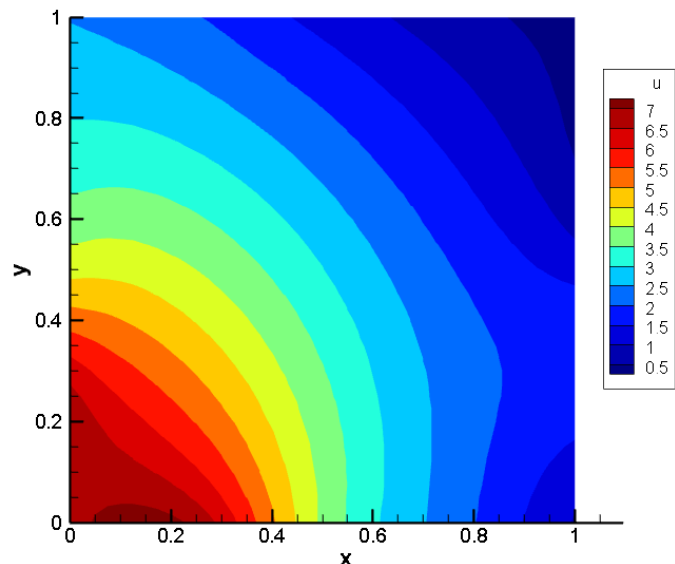


Fig. 10. The contour plot of approximate solution $u(x, y, t)$ with mesh size $\Delta x = \Delta y = 0.025$, $\Delta t = 0.0005$, $t = 0.5$

5. Conclusions

In this paper, the TV-LDG method was proposed to solve the time-dependent linear convection-diffusion equations. The numerical results have proven that the proposed scheme is one of the most accurate methods to approximate this type of PDEs, and the main observations are that the error is decreasing as mesh spacing is decreased as well as the order of the polynomial is increased. And the accuracy of the approximate solution is strongly effective with the type of the diffusion numerical flux. To sum up, the proposed algorithm computationally stable, reliable, effective, and convergent to $(k + 1)$ order of accuracy if polynomials of order k are used.

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