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Properties of Measure Theory and Lebesgue integration

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By

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بسم الله الرحمن الرحيم

قالوا سبحانك لاعلم لنا الا ماعلمتنا انك انت العليم الحكيم

صدق الله العظيم

سورة البقرة الاية 32

Dedication

To my family and my friends and for who wanted the science as a path and life .

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Abstract

In this thesis, we study and investigate the following concepts :

The Lebesgue measure of a set , the class of measurable sets , the class of μ^* - measurable sets , the class of measurable functions and Lebesgue integration .

We give some properties of the above concepts. Also, we give some facts, deductions , different connections , related examples and some applications of Lebesgue integration .

Contents

Chapter One
Preliminaries

In this chapter, we give some definitions and results which we shall need later in this thesis . Also, we give some related examples and remarks .

Notations

$\mathbb{N}$ = the set of natural numbers

$\mathbb{Q}$ = the set of rational numbers

$\mathbb{Q}^c$ = the set of irrational numbers

$\mathbb{R}$ = the set of real numbers .

We start with the basic definitions and results from set Theory .

Definition 1.1

Let A and B be subsets of the universal set X .

The *intersection* of A and B is defined by

$$A \cap B = \{ x : x \in A \text{ and } x \in B \} .$$

Then A and B are called *disjoint* if $A \cap B = \varnothing$.

The *union* of A and B is defined by

$$A \cup B = \{ x : x \in A \text{ or } x \in B \} .$$

Theorem 1.1

Let A, B and C be sets . Then

(i) $A \cap B \subset A$ *and* $A \cap B \subset B$

(ii) $A \subset A \cup B$ *and* $B \subset A \cup B$

(iii) $A \cap B \subset A \subset A \cup B$ *and* $A \cap B \subset B \subset A \cup B$

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(v) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

The intersection of a finite number of sets $A_1 , A_2 , \dots , A_n$ is denoted by

$$A_1 \cap A_2 \cap \dots \cap A_n \quad \text{or} \quad \bigcap_{k=1}^{n} A_k .$$

The intersection of an infinite number of sets $A_1 , A_2 , \dots , A_n , \dots$ is denoted by

$$A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots \quad \text{or} \quad \bigcap_{k=1}^{\infty} A_k .$$

The union of a finite number of sets $A_1, A_2, \ldots, A_n$ is denoted by

$$A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{or} \quad \bigcup_{k=1}^{n} A_k .$$

The union of an infinite number of sets $A_1, A_2, \ldots, A_n, \ldots$ is denoted by

$$A_1 \cup A_2 \cup \ldots \cup A_n \cup \ldots \quad \text{or} \quad \bigcup_{k=1}^{\infty} A_k .$$

Definition 1.2

Let $f : X \to Y$ and let B be a subset of Y. The *inverse of* B *under the mapping* f is defined by

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}.$$

Theorem 1.2

Let $f : X \to Y$ *and let* A *and* B *be subsets of* Y. *Then*

(i) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(iii) $f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(A_i)$

(iv) $f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$.

Definition 1.3

Let $A \subset X$. The *complement* of A is defined by

$$A^c = \{ x : x \in X \text{ and } x \notin A \}.$$

Sometimes, we write $A^c = X \setminus A$.

Theorem 1.3

Let $A, B \subset X$. *Then*

(i) $X^c = \varnothing$, $\varnothing^c = X$

(ii) $(A^c)^c = A$

(iii) $A \cap A^c = \varnothing$, $A \cup A^c = X$

Theorem 1.4

Let $A, B \subset X$ *. If* $A \subset B$ *, then*

(i) $B^c \subset A^c$

(ii) $A \cap B = A$, $A \cup B = B$

(iii) $B \cup A^c = X$, $A \cap B^c = \varnothing$.

Theorem 1.5

Let $f : X \to Y$ *and* $A \subset Y$*. Then*

$$f^{-1}(A^c) = (f^{-1}(A))^c .$$

Theorem 1.6 (De Morgan laws)

Let A *and* B *be sets. Then*

(i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$.

The generalized of Demorgan Laws for any finite number of sets is

(i) $\left(\bigcup_{k=1}^{n} A_k\right)^c = \bigcap_{k=1}^{n} A_k^c$

(ii) $\left(\bigcap_{k=1}^{n} A_k\right)^c = \bigcup_{k=1}^{n} A_k^c$.

The generalized of Demorgan Laws for any infinite number of sets is

(i) $\left(\bigcup_{k=1}^{\infty} A_k\right)^c = \bigcap_{k=1}^{\infty} A_k^c$

(ii) $\left(\bigcap_{k=1}^{\infty} A_k\right)^c = \bigcup_{k=1}^{\infty} A_k^c$.

Definition 1.4

The *difference* of a set A with respect to a set B is defined by

$$A - B = \{ x : x \in A \text{ and } x \notin B \},$$

while the *difference* of a set B with respect to a set A is defined by

$$B - A = \{ x : x \in B \text{ and } x \notin A \}.$$

Sometimes, we write $A - B = A \setminus B$.

Theorem 1.7

Let $A, B \subset X$ *. Then*

(i) $A - B \subset A$, $B - A \subset B$

(ii) $A - B = A \cap B^c$

(iii) If $A \subset B$, then $C \setminus B \subset C \setminus A$

(v) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

More generally, we have

$$\text{(i)} \quad A \setminus \bigcap_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty} (A \setminus B_k)$$

$$\text{(ii)} \quad A \setminus \bigcup_{k=1}^{\infty} B_k = \bigcap_{k=1}^{\infty} (A \setminus B_k).$$

Definition 1.5

Let X be a set. The *power set* of X is the family of all subsets of X.
It is denoted by $P(X)$.
If X contains n elements, then $P(X)$ contains 2^n elements.
Note that $X, \varnothing \in P(X)$.

Examples 1.1

(i) Let $X = \{1\}$. Then

$$P(X) = \{\varnothing, \{1\}\}.$$

(ii) Let $X = \{1,2\}$. Then

$$P(X) = \{\varnothing, \{1\}, \{1,2\}, X\}.$$

(iii) Let $X = \{1,2,3\}$. Then

$$P(X) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}.$$

Definition 1.6

Let X be a non-empty set. Let f be a function from X into $\mathbb{R}$.
The *positive part* of f is defined by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The *negative part* of f is defined by

$$f^{-}(x) = \begin{cases} -f(x) & \text{if } f(x) \le 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remarks 1.1

(i) $f^{+} \ge 0$ and $f^{-} \ge 0$.

(ii) $(f+g)^{+} = f^{+} + g^{+}$

$(f+g)^{-} = f^{-} + g^{-}$.

(iii) Let $\alpha > 0$. Then

$$(\alpha f)^{+} = \alpha f^{+}$$

$$(\alpha f)^{-} = \alpha f^{-}.$$

(iv) Let $\alpha < 0$. Then

$$(\alpha f)^{+} = -\alpha f^{-}$$

$$(\alpha f)^{-} = -\alpha f^{+}.$$

Lemma 1.8

Let X be a non-empty set and let f be a function from X into $\mathbb{R}$. Then

(i) $f = f^{+} - f^{-}$

(ii) $|f| = f^{+} + f^{-}$.

Lemma 1.9

Let $x, y \in \mathbb{R}$ *and let* $\in > 0$ (very small).

(i) *if* $|x - y| < \in$, *then* $x = y$,

(ii) *if* $x \le y + \in$, *then* $x \le y$.

Definition 1.7

Let E be a non-empty subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Then we define

$$E + x = \{ y + x : y \in E \}.$$

Theorem 1.10

Let X *be a non-empty subset of* $\mathbb{R}$ *. Let* E *and* A *be subsets of* X *and* $x \in \mathbb{R}$ *. Then*

(i) *If* $E \subset A$ *, then* $E + x \subset A + x$

(ii) $(A \setminus E) + x = (A + x) \setminus (E + x)$

(iii) $((A - x) \cap E) + x = A \cap (E + x)$

(iv) $((A - x) \cap E^c) + x = A \cap (E + x)^c$.

Definition 1.8

Let A be a non-empty subset of $\mathbb{R}$. An element $x \in \mathbb{R}$ called an *upper bound* of A if $a \le x$ for all $a \in A$.

If A has an upper bound , then A is called a *bounded above set* .

Definition 1.9

Let A be a non-empty subset of $\mathbb{R}$. An element $y \in \mathbb{R}$ called a *lower bound* of A if $y \le a$ for all $a \in A$.

If A has a lower bound , then A is called a *bounded below set* .

Definition 1.10

Let A be a non-empty subset of $\mathbb{R}$. Then A is called a *bounded* if A is both bounded above and bounded below .

Lemma 1.11

Any subset of a bounded set is bounded .

Theorem 1.12

A finite union of bounded sets is bounded .

Remark 1.2

An infinite union of bounded sets may not be bounded .

For example :

Let $A_n = [-n, n]$ ($n = 1, 2, 3, ...$).

Then A_n are bounded sets . We have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} [-n,n]$$

$$= (-\infty,\infty),$$

which is not bounded.

Definition 1.11

Let A be a non-empty subset of $\mathbb{R}$. A real number u is called a *supremum* of A, denoted by $\sup(A)$, if

(i) $a \le u$ for all $a \in A$ (u is an upper bound of A)

(ii) $u \le v$ for any upper bound v of A (u is the least upper bound of A).

If $\sup(A) \in A$, then it is called a *maximum* of A, is denoted by $\max(A)$.

Theorem 1.13

Let A be a non-empty bounded above subset of $\mathbb{R}$. Then $\sup(A)$ exists and unique.

Theorem 1.14

Let A and B be non-empty bounded above subsets of $\mathbb{R}$. If $A \subseteq B$, then $\sup(A) \le \sup(B)$.

Theorem 1.15

Let A be a non- empty bounded above subset of $\mathbb{R}$. Let $\in > 0$ and $\alpha = \sup(A)$. Then there exists $a \in A$ such that $a > \alpha - \in$.

Definition 1.12

Let A be a non-empty subset of $\mathbb{R}$. A real number w is called an *infimum* of A, denoted by $\inf(A)$, if

(i) $w \le a$ for all $a \in A$ (w is a lower bound of A)

(ii) $t \le w$ for any lower bound t of A (w is the greatest lower bound of A).

If $\inf(A) \in A$, then it is called a *minimum* of A, is denoted by $\min(A)$.

Theorem 1.16

Let A be a non-empty bounded below subset of $\mathbb{R}$. Then $\inf(A)$ exists and unique.

Theorem 1.17

Let A *and* B *be non-empty bounded below subsets of* $\mathbb{R}$. *If* $A \subseteq B$, *then* $\inf(B) \leq \inf(A)$.

Theorem 1.18

Let A *be a non- empty bounded below subset of* $\mathbb{R}$. *Let* $\in > 0$ *and* $\beta = \inf(A)$. *Then there exists* $a \in A$ *such that* $a < \beta + \in$.

Definition 1.13

Let X be a bounded set. A mapping $f : X \rightarrow \mathbb{R}$ is called *bounded* if there exists a positive real number M such that

$$|f(x)| \leq M \quad \text{for all } x \in X.$$

Example 1.2

Let $f(x) = 3x + 4$, $X = [-2, 2]$.

Then X is a bounded set .

Let $x \in [-2, 2]$. Then $|x| \leq 2$.

So
$$\begin{aligned} |f(x)| &= |3x + 4| \\ &\leq 3|x| + 4 \\ &\leq 3(2) + 4 \\ &= 10. \end{aligned}$$

Thus f is a bounded function on X with $M = 10$.

Theorem 1.19

Let X *be a bounded subset of* $\mathbb{R}$ *and let* $f : X \rightarrow \mathbb{R}$ *be a bounded function* . *Then*

(i) $\sup_{x \in X} (\alpha f(x)) = \alpha \sup_{x \in X} (f(x)) \quad (\alpha > 0)$

(ii) $\sup_{x \in X} (\alpha f(x)) = \alpha \inf_{x \in X} (f(x)) \quad (\alpha < 0)$.

Definition 1.14

Let X be a non – empty set . Let d be a function defined on the cartesian product $X \times X$ into $\mathbb{R}$ such that

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$. Then d is called a *metric* on X and (X, d) is called a *metric space* .

Example 1.3

Let $X = \mathbb{R}$. Define d by

$$d(x, y) = |x - y| \quad (x, y \in X).$$

Then d is a metric on X and (X, d) is a metric space .

This metric space is called the *usual metric space* .

Definition 1.15

Let (X, d) be a metric space and $x \in X$ and Let $r > 0$. The set

$$B(x, r) = \{ y \in X : d(x, y) < r \}.$$

is called an *open ball* with center x and radius r .

Definition 1.16

Let (X, d) be a metric space . A subset A of X is said to be *open* in X if for each $x \in A$ there is $r > 0$ such that $B(x, r) \subseteq A$.

Definition 1.17

A subset A of a metric space (X, d) is called *a closed set* in (X, d) if its complement A^c is open in (X, d).

Examples 1.4

Let $(\mathbb{R}, d)$ be the *usual metric space* .

(i) The empty set $\varnothing$ and the universal set $\mathbb{R}$ are open and closed .

(ii) Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Then

$$\mathbb{N}^c = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup \ldots$$

So $\mathbb{N}^c$ is open and hence $\mathbb{N}$ is a closed set .

(iii) Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\}$. Then

$$A^c = (-\infty, 1) \cup (1, \tfrac{1}{2}) \cup (\tfrac{1}{2}, \tfrac{1}{3}) \cup \ldots \cup (0, \infty).$$

So A^c is open and hence A is a closed set .

(iv) $\mathbb{Q}$ is neither open nor closed .

Also, $\mathbb{Q}^c$ is neither open nor closed .

(v) Let $A = (1, 3) \cup \{5\}$.

Then A is neither open nor closed .

Theorem 1.20

(i) *The intersection of any finite number of open sets in a metric space* (X, d) *is open* .

(ii) *The union of any collection of open sets* (*finite or infinite*) *in a metric space* (X, d) *is open* .

Remark 1.3

An infinite intersection of open sets may not be open .

For example :

Let $A_n = (-\frac{1}{n}, \frac{1}{n})$ $(n = 1, 2, 3, \ldots)$.

Then A_n are open sets . We have

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$$

$$= \{0\},$$

which is not open .

Theorem 1.21

(i) *The intersection of any collection of closed sets* (*finite or infinite*) *in a metric space* (X, d) *is closed* .

(ii) *The union of any finite number of closed sets in a metric space* (X, d) *is closed* .

Remark 1.4

An infinite union of closed sets may not be closed .

For example :

Let $F_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ $(n = 1, 2, 3, \ldots)$.

Then F_n are closed sets . We have

$$\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1 - \frac{1}{n}]$$

$$= (0, 1),$$

which is not closed .

Definition 1.18

Let (X, d) and (Y, d) be two metric spaces . A function $f : (X, d) \to (Y, d)$ is called *continuous at* x_0 in X if for each $\in > 0$ there exists $\delta > 0$ such that

$$d(f(x), f(x_0)) < \in \quad \text{for all} \quad d(x, x_0) < \delta.$$

The function f is called *continuous on* X if it is continuous at each point of X .

Examples 1.5

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2x + 1.$$

Let $x, x_0 \in \mathbb{R}$. Then

$$|f(x) - f(x_0)| = |(2x + 1) - (2x_0 + 1)|$$

$$= 2|x - x_0|.$$

Thus if $|x - x_0| < \delta$, it follows that

$$|f(x) - f(x_0)| < 2\delta.$$

Choose $\delta = \frac{\in}{2}$. Therefore

$$|f(x) - f(x_0)| < \in.$$

Hence f is continuous on $\mathbb{R}$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sin x .$$

Let $x, x_0 \in \mathbb{R}$. Then

$$| f(x) - f(x_0) | = | \sin x - \sin x_0 |$$

$$\leq | x - x_0 |.$$

Thus if $| x - x_0 | < \delta$, it follows that

$$| f(x) - f(x_0) | < \delta .$$

Choose $\delta = \in$. Therefore

$$| f(x) - f(x_0) | < \in.$$

Hence f is continuous on $\mathbb{R}$.

Theorem 1.22

Let (X, d) and (Y, d) be two metric spaces. Let $f, g : (X, d) \to (Y, d)$ be continuous functions. Then

$$f + g,\ f - g,\ \alpha f,\ f g,\ \frac{f}{g}\ (g \neq 0)$$

are continuous functions.

Theorem 1.23

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}(O)$ is open for every open O in $\mathbb{R}$.

Definition 1.19

Let X be a non-empty set whose elements are called vectors and let K be the field of scalars and in which two operations called addition and scalar multiplication are defined. Then X is called a *linear space* (or *a vector space*) over K if for all $x, y, z \in X$ and $\alpha, \beta \in K$ the following axioms hold :

(i) $(x + y) + z = x + (y + z)$.

(ii) $x + y = y + x$.

(iii) There exists 0 in X such that

$$x + 0 = x = 0 + x,$$

(0 is called the *zero vector*) .

(iv) There exists $-x$ in X such that

$$x + (-x) = 0 = (-x) + x ,$$

($-x$ is called the *additive inverse* of x) .

(v) $\alpha\,(x + y) = \alpha x + \alpha y$.

(vi) $(\alpha + \beta)x = \alpha x + \beta x$.

(vii) $(\alpha\beta)x = \alpha(\beta x)$.

(viii) $1 \cdot x = x$,

(1 is called the *multiplicative identity*) .

Examples 1.6 [5]

(i) Let $\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R} \}$

= n - Euclidean space .

The addition on $\mathbb{R}^n$ is given by :

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

The scalar multiplication on $\mathbb{R}^n$ is given by :

$$\alpha\,(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n) ,$$

for all $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Then $\mathbb{R}^n$ is a linear space over $\mathbb{R}$.

(ii) Let X be the set of all polynomials

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n ,$$

with coefficients a_i ($i = 1, 2, \ldots, n$) from a field K .

Then X is a linear space over K with respect to the usual operations of addition of polynomials and multiplication by a constant .

(iii) Let X be the set of all $m \times n$ matrices with entries from an arbitrary field K .

Then X is a linear space over K with respect to the the operations of matrix addition and multiplication by a constant .

Definition 1.20

Let X, Y be linear spaces over the same field K. A mapping f from X into Y is called *linear* if

$$\text{(i) } f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in X,$$

$$\text{(ii) } f(\alpha x) = \alpha f(x) \quad \text{for all } x \in X, \alpha \in K,$$

or f is called a *linear mapping* if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad (x, y \in X, \alpha, \beta \in K).$$

Examples 1.7

(i) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = 2x - 3y + 4z.$$

Let $u = (a_1, b_1, c_1)$ and $v = (a_2, b_2, c_2)$.

Let $\alpha \in \mathbb{R}$. Then

$$\alpha u = \alpha(a_1, b_1, c_1)$$

$$= (\alpha a_1, \alpha b_1, \alpha c_1).$$

Therefore

$$f(\alpha u) = f(\alpha a_1, \alpha b_1, \alpha c_1)$$

$$= 2\alpha a_1 - 3\alpha b_1 + 4\alpha c_1$$

$$= \alpha(2a_1 - 3b_1 + 4c_1)$$

$$= \alpha f(u),$$

and we have

$$f(u+v) = f(a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

$$= 2(a_1 + a_2) - 3(b_1 + b_2) + 4(c_1 + c_2)$$

$$= (2a_1 - 3b_1 + 4c_1) + (2a_2 - 3b_2 + 4c_2)$$

$$= f(u) + f(v).$$

Thus f is a linear mapping.

(ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = xy.$$

Let $u = (a, b)$. Then

$$f(u) = f(a, b)$$
$$= ab,$$

and we have

$$f(\alpha u) = f(\alpha(a, b))$$
$$= f(\alpha a, \alpha b)$$
$$= (\alpha a)(\alpha b)$$
$$= \alpha^2 ab$$
$$\neq \alpha f(u).$$

Thus f is not a linear mapping.

Definition 1.21

Let (f_n) be a sequence of functions defined on X. Then for each $x \in X$, we define the *limit superior* and the *limit inferior* by

$$\liminf_{n \to \infty} (f_n(x)) = \lim_{n \to \infty} \inf \{ f_k(x) : k \geq n \}$$
$$= \sup_n \inf \{ f_k(x) : k \geq n \}$$

and

$$\limsup_{n \to \infty} (f_n(x)) = \lim_{n \to \infty} \sup \{ f_k(x) : k \geq n \}$$
$$= \inf_n \sup \{ f_k(x) : k \geq n \}.$$

Notation

$$\liminf_{n \to \infty} (f_n(x)) = \underline{\lim}\, f_n(x)$$

$$\limsup_{n \to \infty} (f_n(x)) = \overline{\lim}\, f_n(x).$$

Examples 1.8

(i) Define $f_n : \mathbb{R} \to [-1, 1]$ by

$$f_n(x) = \sin nx.$$

Then

$$\liminf_{n \to \infty} f_n(x) = -1,$$

and

$$\limsup_{n \to \infty} f_n(x) = 1.$$

(ii) Define the sequence of functions (f_n) by

$$f_n(x) = \begin{cases} 1 & \text{if } n \text{ is even} \\ -\dfrac{1}{n} & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\liminf_{n \to \infty} f_n(x) = 0,$$

and

$$\limsup_{n \to \infty} f_n(x) = 1.$$

Theorem 1.24

Let (f_n) be a sequence of functions defined on X and $x \in X$. Then

(i) $\liminf_{n \to \infty} (f_n(x)) \le \limsup_{n \to \infty} (f_n(x))$

(ii) $\liminf_{n \to \infty} (-f_n(x)) = -\limsup_{n \to \infty} (f_n(x))$.

Theorem 1.25

Let (f_n) be a sequence of functions defined on X and $x \in X$. If $f(x) = \lim_{n \to \infty} f_n(x)$, *then*

$$f(x) = \limsup_{n \to \infty} f_n(x) = \liminf_{n \to \infty} f_n(x).$$

Example 1.9

Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{x^2}{1 + nx^2}.$$

Then $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\frac{x^2}{n}}{\frac{1}{n} + x^2}$

$$= 0.$$

It follows from Theorem 1.25 that

$$\liminf_{n \to \infty} f_n(x) = 0,$$

and

$$\limsup_{n \to \infty} f_n(x) = 0.$$

Definition 1.22

Let X be a non – empty set. A non - empty family F of subsets of X is called a *field* if

(i) $X, \emptyset \in F$

(ii) for each $A \in F$, then $A^c \in F$

(iii) If $A_1, A_2, \ldots, A_n \in F$, then $\bigcup_{k=1}^{n} A_k \in F$.

Examples 1.10

(i) Let X be any set and let $F = \{\emptyset, X\}$.

Then F is a field (the smallest field of X).

(ii) Let $X = \{1, 2, 3\}$.

Let $F = \{\emptyset, X, \{1\}, \{2, 3\}\}$.

Then F is a field.

(iii) Let $X = [0, 1]$.

Let $F = \{\emptyset, X, [0, \frac{1}{2}], (\frac{1}{2}, 1]\}$.

Then F is a field.

(iv) Let $X = \mathbb{N}$ = the set of all natural numbers.

Let $F = \{\varnothing, \mathbb{N}, \{1\}, \{2\}, \{1,2\}, \mathbb{N}\setminus\{1\}, \mathbb{N}\setminus\{2\}, \mathbb{N}\setminus\{1,2\}\}$.

Then F is a field.

Lemma 1.26

Let F be a field of subsets of X and let $A, B \in F$. Then

$$A - B \in F.$$

Lemma 1.27

Let F be a field of subsets of X. If $A_1, A_2, \dots, A_n \in F$, then $\bigcap_{k=1}^{n} A_k \in F$.

Remark 1.5

Let F_1 and F_2 be two fields of subsets of X. Then $F_1 \cup F_2$ may not be a field.

For example:

Let $X = \{1, 2, 3\}$.

Let $F_1 = \{\varnothing, X, \{1\}, \{2, 3\}\}$,

$F_2 = \{\varnothing, X, \{2\}, \{1, 3\}\}$.

Then F_1 and F_2 are fields of subsets of X.

We have

$$F_1 \cup F_2 = \{\varnothing, X, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}.$$

Thus $F_1 \cup F_2$ is not a field of subsets of X.

Definition 1.23

Let X be a non-empty set. A non-empty family F of subsets of X is called a $\sigma-$field if

(i) $X, \varnothing \in F$

(ii) for each $A \in F$, then $A^c \in F$

(iii) If A_k $(k \in \mathbb{N}) \in F$, then $\bigcup_{k=1}^{\infty} A_k \in F$.

Examples 1.11

(i) Let X be a non-empty set and let $F = \{\varnothing, X\}$.

Then F is a σ-field (the smallest σ-field of X).

(ii) Let X be the set of all real numbers. Let $F = P(X)$.

Then F is a σ - field (the largest σ - field of X).

Remark 1.6

Every σ - field is a field . In general, the converse is not true .

For example :

Let $X = (0, 1]$.

Let F be the class consisting of $\varnothing$ and of all finite disjoint unions of the form

$$A = \bigcup_{i=1}^{n} (a_i, b_i] \quad (0 < a_i \le b_i \le 1).$$

We have

(i) $X, \varnothing \in F$.

(ii) Let $A \in F$. Then

$$A^c = (0, a_1] \cup (b_1, a_2] \cup \ldots \cup (b_n, 1] \in F .$$

(iii) Let $(a, b], (c, d] \in F$. Then

$$(a, b] \cup (c, d] \in F .$$

Thus F is a field .

Let $A_n = (0, 1 - \frac{1}{n}] \in F$.

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (0, 1 - \frac{1}{n}]$

$$= (0, 1) \notin F .$$

Thus F is not a σ - field .

Lemma 1.28

Let F be a σ - field of subsets of X and let $A, B \in F$. Then

$$A - B \in F .$$

Lemma 1.29

Let F be a σ - field of subsets of X . If A_n $(n \in \mathbb{N}) \in F$, then $\bigcap_{n=1}^{\infty} A_n \in F$.

Definition 1.24

Let $A \subset X$. Then the real-valued function $\chi_A : X \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in A^c, \end{cases}$$

is called the *characteristic function* of A .

Example 1.12

Let $X = \mathbb{N}$ and let $A = \{1, 2, 3, 4\}$.

Then $\chi_A(1) = \chi_A(2) = \chi_A(3) = \chi_A(4) = 1$,

while , for examples

$$\chi_A(5) = 0, \ \chi_A(6) = 0, \ \chi_A(7) = 0.$$

Some properties of characteristic functions

Let $A, B \subset X$. Then

(i) $\chi_{\varnothing} = 0$

(ii) If $A \subseteq B$, then $\chi_A \leq \chi_B$

(iii) $\chi_{A^C} = 1 - \chi_A$

(iv) $\chi_{A \cap B} = \chi_A \cdot \chi_B$

(v) $\chi_{A \setminus B} = \chi_A - \chi_{A \cap B}$

(vi) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}$.

Remark 1.7

If $A \cap B = \phi$, then (vi) becomes

$$\chi_{A \cup B} = \chi_A + \chi_B .$$

More generally , if $n \in \mathbb{N}$ and $A_1 \cap A_2 \cap \ldots \cap A_n = \phi$, then we have

$$\chi_{A_1 \cup A_2 \cup \ldots \cup A_n} = \chi_{A_1} + \chi_{A_2} + \cdots + \chi_{A_n} .$$

Definition 1.25

Let $A \subset X$. A *simple function* is a function $\phi : X \to \mathbb{R}$ of the form

$$\phi(x) = \sum_{i=1}^{n} a_i \, \chi_{A_i}(x),$$

where $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and χ_{A_i} are the characteristic functions of A.

Theorem 1.30

Let ϕ_1, ϕ_2 be simple functions. Then $\phi_1 + \phi_2$ is a simple function.

The following theorem is a generalization of Theorem 1.30

Theorem 1.31

Let $n \in \mathbb{N}$ and let $\phi_1, \phi_2, \ldots, \phi_n$ be simple functions. Then $\phi_1 + \phi_2 + \ldots + \phi_n$ is a simple function.

Lemma 1.32

Let ϕ be a simple function and let α be a constant. Then $\alpha\phi$ is a simple function.

The next corollary follows from Theorem 1.31 and Lemma 1.32.

Corollary 1.33

Let $n \in \mathbb{N}$ and let $\phi_1, \phi_2, \ldots, \phi_n$ be simple functions. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be constants. Then $\alpha_1 \phi_1 + \alpha_2 \phi_2 + \ldots + \alpha_n \phi_n$ is a simple function.

Chapter Two

Properties of the Lebesgue measure of a set

In this chapter, we give some properties of the Lebesgue measure of open and closed sets. Also, we give some properties of the Lebesgue exterior measure and the Lebesgue interior measure.

2.1 The Lebesgue measure of open and closed sets

The length of an infinite interval such as (a, ∞) or $(-\infty, b)$ of $\mathbb{R}$ defined to be ∞ while the length of a bounded interval of $\mathbb{R}$ is defined to be the difference between two end points. We begin with the measure of a bounded interval of $\mathbb{R}$ which agree with the idea of length.

Definition 2.1.1

Let $I = (a, b)$ or $\big((a, b], [a, b), [a, b] \big)$ be a bounded subset of $\mathbb{R}$. We define the *measure* (*the Lebesgue measure*) or length of I by

$$m(I) = b - a.$$

Remark 2.1.1

It is clear that $0 \le m(I) < \infty$. That is, the measure of a bounded interval I of $\mathbb{R}$ is a non-negative real number.

Examples 2.1.1

(i) $m\big((-\frac{1}{2}, \frac{1}{2})\big) = \frac{1}{2} - (-\frac{1}{2}) = 1.$

(ii) $m([2, \frac{5}{2})) = \frac{5}{2} - 2 = \frac{1}{2}.$

(iii) $m\big(\{x : \sqrt{2} < x \le \sqrt{3}\}\big) = m\big((\sqrt{2}, \sqrt{3}]\big)$

$$= \sqrt{3} - \sqrt{2}.$$

(iv) Let $S = [-1, 1) \cup (0, 2].$

Then $S = [-1, 2]$.

So $m(S) = m([-1, 2])$

$= 2 - (-1)$

$= 3$.

The next lemma gives us some sets which have measure zero .

Lemma 2.1.1

(i) *The measure of an empty set* $\varnothing$ *is zero. That is ,* $m(\varnothing) = 0$.

(ii) *If* A *is a singleton set, then* $m(A) = 0$.

Proof

(i) We have $\varnothing = (a, a) = (a, a] = [a, a)$.

So

$$m(\varnothing) = m\big((a, a)\big) = a - a = 0.$$

(ii) Let A be a singleton set . Then $A = \{ a \}$ $(a \in A)$.

We have $\{ a \} = [a, a]$.

Therefore $m(\{ a \}) = m([a, a])$

$= a - a$

$= 0$.

Definition 2.1.2

Let S be a non-empty set such that $S = \bigcup_{i=1}^{n} I_i$, where $I_1, I_2, \ldots, I_n$ are pairwise disjoint intervals . We define the *measure* of S by

$$m(S) = m\Big(\bigcup_{i=1}^{n} I_i\Big)$$

$$= m(I_1) + m(I_2) + \ldots + m(I_n)$$

$$= \sum_{i=1}^{n} m(I_i).$$

Remark 2.1.2

It is clear that $0 \le m(S) < \infty$.

Examples 2.1.2

(i) Let $S = [\frac{1}{3}, \frac{1}{2}) \cup [\frac{1}{2}, 1)$.

Then

$$m(S) = m\left([\tfrac{1}{3}, \tfrac{1}{2}) \cup [\tfrac{1}{2}, 1)\right)$$

$$= m([\tfrac{1}{3}, \tfrac{1}{2})) + m([\tfrac{1}{2}, 1))$$

$$= (\tfrac{1}{2} - \tfrac{1}{3}) + (1 - \tfrac{1}{2})$$

$$= \tfrac{2}{3}.$$

(ii) Let $S = (-2, -1) \cup (0, 1) \cup (2, 4)$.

Then

$$m(S) = m((-2, -1) \cup (0, 1) \cup (2, 4))$$

$$= m((-2, -1)) + m((0, 1)) + m((2, 4))$$

$$= (-1 + 2) + (1 - 0) + (4 - 2)$$

$$= 4.$$

(iii) Let $S = \{x \in \mathbb{R} : 4 \le x^2 \le 9\}$.

Then

$$S = [-3, -2] \cup [2, 3].$$

So

$$m(S) = m\left([-3, -2] \cup [2, 3]\right)$$

$$= m([-3, -2]) + m([2, 3])$$

$$= (-2 - (-3)) + (3 - 2)$$

$$= 2.$$

In the next definition, we extend the idea of the measure of an open interval to the measure of an open set .

Definition 2.1.3

Let G be a non-empty bounded open set of real numbers such that

$$G = \bigcup_{i=1}^{\infty} I_i ,$$

where I_i are pairwise disjoint open intervals .

The *measure* of G is defined by

$$m(G) = m\left(\bigcup_{i=1}^{\infty} I_i \right) = \sum_{i=1}^{\infty} m(I_i).$$

Remark 2.1.3

It is clear that $0 \leq m(G) < \infty$.

Example 2.1.3

Let $G = \bigcup_{k=1}^{\infty} \{ x : \frac{3}{2^{k+1}} < x < \frac{1}{2^{k-1}} \}$.

Then G is a bounded open subset of $(0,1)$.

We have

$$I_1 = \frac{3}{4} < x < 1$$

$$I_2 = \frac{3}{8} < x < \frac{1}{2}$$

$$I_3 = \frac{3}{16} < x < \frac{1}{4}.$$

In the same way , we can get

$$I_k = \frac{3}{2^{k+1}} < x < \frac{1}{2^{k-1}}.$$

So we have

$$m(I_1) = 1 - \frac{3}{4}$$

$$= \frac{1}{2}\left(2 - \frac{3}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{2}$$

$$m(I_2) = \frac{1}{2} - \frac{3}{8}$$

$$= \frac{1}{4}\left(2 - \frac{3}{2}\right)$$

$$= \frac{1}{4} \cdot \frac{1}{2}$$

$$m(I_3) = \frac{1}{4} - \frac{3}{16}$$

$$= \frac{1}{8}\left(2 - \frac{3}{2}\right)$$

$$= \frac{1}{8} \cdot \frac{1}{2},$$

and so we have

$$m(I_n) = \frac{1}{2^{n-1}} - \frac{3}{2^{n+1}}$$

$$= \frac{1}{2^n}\left(2 - \frac{3}{2}\right)$$

$$= \frac{1}{2^n} \cdot \frac{1}{2}.$$

Thus

$$m(G) = m\left(\bigcup_{k=1}^{\infty} I_k\right)$$

$$= \sum_{k=1}^{\infty} m(I_k)$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} m(I_k)$$

$$= \lim_{n \to \infty} \left(m(I_1) + m(I_2) + \ldots + m(I_n)\right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{8} + \ldots + \frac{1}{2} \cdot \frac{1}{2^n} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{2} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \right) \right)$$

$$= \frac{1}{2} \lim_{n \to \infty} \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \right)$$

$$= \frac{1}{2} \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{2} \right)^k$$

$$= \frac{1}{2} \left(\frac{\frac{1}{2}}{1 - \frac{1}{2}} \right)$$

$$= \frac{1}{2}.$$

Thus $m(G) = \frac{1}{2}$.

Theorem 2.1.2

Let G_1 and G_2 be disjoint bounded open sets. Then

$$m(G_1 \cup G_2) = m(G_1) + m(G_2).$$

Proof

Let G_1 be a bounded open set. Then

$$G_1 = \bigcup_{i=1}^{\infty} I_i, \text{ where } I_i \cap I_j = \varnothing, i \neq j.$$

Let G_2 be a bounded open set. Then

$$G_2 = \bigcup_{i=1}^{\infty} I'_i, \text{ where } I'_i \cap I'_j = \varnothing, i \neq j,$$

where G_1 and G_2 are disjoint bounded open sets and I_i, I'_i are pairwise disjoint open intervals. Then

$$G_1 \cup G_2 = \left(\bigcup_{i=1}^{\infty} I_i \right) \cup \left(\bigcup_{i=1}^{\infty} I'_i \right).$$

We have

$$m(G_1 \cup G_2) = m\left((\bigcup_{i=1}^{\infty} I_i) \cup (\bigcup_{i=1}^{\infty} I'_i) \right)$$

$$= m\left(\bigcup_{i=1}^{\infty} (I_i \cup I'_i) \right)$$

$$= \sum_{i=1}^{\infty} m\left(I_i \cup I'_i \right)$$

$$= \sum_{i=1}^{\infty} \left(m(I_i) + m(I'_i) \right)$$

$$= \sum_{i=1}^{\infty} m(I_i) + \sum_{i=1}^{\infty} m(I'_i)$$

$$= m(G_1) + m(G_2).$$

Hence

$$m(G_1 \cup G_2) = m(G_1) + m(G_2).$$

Theorem 2.1.3

Let $G_1, G_2, \ldots, G_n$ *be disjoint bounded open sets. Then*

$$m(\bigcup_{i=1}^{n} G_i) = \sum_{i=1}^{n} m\left(G_i\right).$$

Proof

We use mathematical induction .

Let $n = 1$. Then $m(G_1) = m(G_1)$ is true .

Let $n = k$. Then

$$m\left(\bigcup_{i=1}^{k} G_i \right) = \sum_{i=1}^{k} m\left(G_i\right)$$

$$= m(G_1) + m(G_1) + \ldots + m(G_k).$$

We will show that it is true for $n = k + 1$.

We have

$$m\left(\bigcup_{i=1}^{k+1} G_i \right) = m\left((G_1 \cup G_2 \cup \ldots \cup G_k) \cup G_{k+1} \right)$$

$$= m(G_1 \cup G_2 \cup ... \cup G_k) + m(G_{k+1}) \text{ (Theorem 2.1.2)}$$

$$= m(G_1) + m(G_1) + ... + m(G_k) + m(G_{k+1})$$

$$= \sum_{i=1}^{k+1} m(G_i).$$

Hence it is true for n . That is , we have

$$m(\bigcup_{i=1}^{n} G_i) = \sum_{i=1}^{n} m(G_i).$$

Theorem 2.1.4

Let $G_1, G_2, ...$ *and* $\bigcup_{n=1}^{\infty} G_n$ *be bounded sets . Let* $G_1, G_2, ...$ *be disjoint open sets. Then*

$$m(\bigcup_{i=1}^{\infty} G_i) = \sum_{i=1}^{\infty} m(G_i).$$

Proof

Let $G_n = \bigcup_{i=1}^{\infty} I_i^n$, where $\{ I_i^n \}$ is the family of pairwise disjoint open intervals of G_n . Then

$$m(\bigcup_{n=1}^{\infty} G_n) = m\left(\bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{\infty} I_i^n\right)\right)$$

$$= \sum_{n=1}^{\infty} m(\bigcup_{i=1}^{\infty} I_i^n)$$

$$= \sum_{n=1}^{\infty} m(G_n).$$

Theorem 2.1.5 [2]

Let G_1 *and* G_2 *be bounded open sets and* $G_1 \subset G_2$. *Then*

(i) $m(G_1) \le m(G_2)$

(ii) $m(G_2 - G_1) = m(G_2) - m(G_1)$.

Remark 2.1.4

Let G be a bounded open set in $[a, b]$. Then

$$m(G) \le b - a.$$

Theorem 2.1.6 [2]

Let G_1 and G_2 be bounded open sets. Then

$$m(G_1 \cup G_2) = m(G_1) + m(G_2) - m(G_1 \cap G_2).$$

Theorem 2.1.7 [2]

Let $G_1, G_2, \ldots$ and $\bigcup_{n=1}^{\infty} G_n$ be bounded sets. Let $G_1, G_2, \ldots$ be open sets. Then

$$m\left(\bigcup_{i=1}^{\infty} G_i\right) \le \sum_{i=1}^{\infty} m(G_i).$$

Lemma 2.1.8

Let I be a bounded open interval and $a \in \mathbb{R}$. Then

$$m(I + a) = m(I).$$

Proof

Let $I = (A, B)$ and $a \in \mathbb{R}$. Then

$$\begin{aligned} I + a &= (A, B) + a \\ &= (A + a, B + a). \end{aligned}$$

Trerefore

$$\begin{aligned} m(I + a) &= m((A + a, B + a)) \\ &= (B + a) - (A + a) \\ &= B - A \\ &= m(I). \end{aligned}$$

Theorem 2.1.9

Let G be a bounded open set and $a \in \mathbb{R}$. Then

$$m(G + a) = m(G).$$

Proof

Let G be a bounded open set. Then

$$G = \bigcup_{i=1}^{\infty} I_i,$$

where I_i are pairwise disjoint open intervals.

Let $a \in \mathbb{R}$. Then $G + a$ is a bounded open set.

So we have

$$m(G+a) = \sum_{i=1}^{\infty} m(I_i + a)$$
$$= \sum_{i=1}^{\infty} m(I_i) \quad (\text{Lemma } 2.1.8)$$
$$= m(G).$$

Definition 2.1.4

Let F be a non-empty closed set contained in $[a, b]$. We define the *measure* of F by

$$m(F) = (b-a) - m(F^c),$$

where $F^c = [a, b] \setminus F$.

Remarks 2.1.5

(i) Note that, if F is a non-empty closed set contained in $[a, b]$, then

$$0 \le m(F) < \infty.$$

(ii) It follows from Definition 2.1.4 that

$$m([a, b] \setminus F) = m([a, b]) - m(F).$$

Examples 2.1.4

(i) Let $F = [3, 5]$ be a closed set contained in $[1, 7]$.

Then

$$F^c = (1, 3) \cup (5, 7).$$

So

$$m(F) = (b-a) - m(F^c)$$
$$= (7-1) - m((1,3) \cup (5,7))$$
$$= (7-1) - (m((1,3)) + m((5,7)))$$
$$= (7-1) - ((3-1) + (7-5))$$
$$= 2.$$

(ii) Let $F = [\,0, 1\,]$ be a closed set contained in $[-1, 1]$.

Then

$$m(F) = (b - a) - m(F^c)$$

$$= (1 - (-1)) - m((-1, 0))$$

$$= (1 - (-1)) - (0 - (-1))$$

$$= 1.$$

Lemma 2.1.10 [3]

Le F be a closed subset of an open set G of $[a, b]$. Then

$$m(F) \le m(G).$$

For the next lemma , we give another method of the proof .

Lemma 2.1.11

Let $F_1, F_2 \subset [a, b]$. Let F_1 be a closed subset of a closed set F_2. Then

$$m(F_1) \le m(F_2).$$

Proof

Let F_1, F_2 be closed sets in $[a, b]$. Then $[a, b] \setminus F_1$ and $[a, b] \setminus F_2$ are open . Since $F_1 \subset F_2$, so $[a, b] \setminus F_2 \subset [a, b] \setminus F_1$.

Then

$$m([a, b] \setminus F_2) \le m([a, b] \setminus F_1) \quad (\text{ Theorem } 2.1.5 \text{ (i)}),$$

and hence by Remark 2.1.5 (ii) , we get

$$m([a, b]) - m(F_2) \le m([a, b]) - m(F_1).$$

So

$$b - a - m(F_2) \le b - a - m(F_1).$$

It follows that

$$-m(F_2) \le -m(F_1).$$

Hence

$$m(F_1) \le m(F_2).$$

Lemma 2.1.12

Let G be an open subset of a closed set F of $[a, b]$. Then

$$m(G) \leq m(F).$$

Proof

Let G be an open subset of a closed set F of $[a, b]$. Then G and $[a, b] \setminus F$ are open and disjoint sets. So $G \cup ([a, b] \setminus F)$ is open .

We have

$$G \cup ([a, b] \setminus F) \subset (a, b).$$

Therefore

$$m(G \cup ([a, b] \setminus F)) \leq m((a, b)) \text{ (Theorem 2.1.5 (i)) }.$$

So

$$m(G) + m([a, b] \setminus F) \leq m((a, b)) \text{ (Theorem 2.1.2) }.$$

Since $m([a, b] \setminus F) = m([a, b]) - m(F)$, it follows that

$$m(G) + m([a, b]) - m(F) \leq m((a, b)).$$

We have

$$m([a, b]) = m((a, b)) = b - a.$$

It follows that

$$m(G) + (b - a) - m(F) \leq b - a.$$

Hence

$$m(G) \leq m(F).$$

2.2 The Lebesgue exterior measure

If E is an open set or closed set , then we have defined its measure as sum of lengths of intervals . But if E is neither open or closed , we can not define its measure by the above method . However, we can define its exterior measure as follows :

Definition 2.2.1

Let $E \subset [a, b]$. We define the *Lebesgue exterior measure* or simply *exterior measure* of E, denoted by $m^*(E)$ by :

$$m^*(E) = \inf \{ m(G) : G \text{ is open and } E \subset G \}.$$

Remarks 2.2.1

(i) Let G be an open set and $E \subset G$. Then

$$m^*(E) \le m(G).$$

(ii) Let G be a bounded open set in $[a, b]$. Then

$$m(G) \le b - a.$$

It follows from (i) that

$$0 \le m^*(E) \le b - a.$$

Hence $m^*(E)$ is finite and exists .

Example 2.2.1

Let $E = Q \cap [0, 1]$

$=$ the set of all rational numbers between 0 and 1.

Let $\in > 0$ and let $\{ q_i : i \in N \}$ be the set of points of E. Then there is an open interval of length $\frac{\in}{2}$ contains q_1 and there is an open interval of length $\frac{\in}{4}$ contains q_2. In general, there is an open interval of length $\frac{\in}{2^n}$ contains q_n.

We have $E \subset \bigcup_{i=1}^{\infty} I_i$ and $\bigcup_{i=1}^{\infty} I_i$ is open .

It follows that

$$\begin{aligned} m^*(E) &\le m\left(\bigcup_{i=1}^{\infty} I_i\right) \quad \text{(By Remark 2.2.1 (i))} \\ &= \sum_{i=1}^{\infty} m(I_i) \\ &= \frac{\in}{2^1} + \frac{\in}{2^2} + \frac{\in}{2^4} + \ldots \\ &= \in \sum_{n=1}^{\infty} \frac{1}{2^n} \end{aligned}$$

$$= \in .$$

Thus $m^*(E) \le \in$.

Since $\in$ is an arbitrary positive number, so

$$m^*(E) = 0.$$

Lemma 2.2.1

Let a be a real number . Then $m^*(\{a\}) = 0$.

Proof

Let $\in > 0$. Then

$$\{a\} \subseteq (a - \in, \ a + \in).$$

Thus

$$m^*(\{a\}) \le m(a - \in, \ a + \in)$$

$$= (a + \in) - (a - \in)$$

$$= 2\in.$$

Since $\in$ is an arbitrary positive number, so

$$m^*(\{a\}) = 0.$$

Theorem 2.2.2

If E is an open set , then $m^*(E) = m(E)$.

Proof

Let E be an open set . Then

$$m^*(E) \le m(E) \quad \rightarrow (\text{i})$$

Let G be open and $E \subseteq G$. Then

$$m(E) \le m(G) \quad (\text{ Theorem 2.1.5 (i)}).$$

Taking infimum of both sides over $E \subseteq G$. Then we have

$$m(E) \le \inf \{ m(G) : G \text{ is open and } E \subset G \}.$$

Thus

$$m(E) \le m^*(E) \quad \rightarrow (\text{ii})$$

It follows from (i) and (ii) that

$$m^*(E) = m(E).$$

Examples 2.2.2

(i) Since $\varnothing$ is an open set , it follows from Theorem 2.2.2 that

$$m^*(\varnothing) = m(\varnothing).$$

We have $m(\varnothing) = 0$ (Lemma 2.1.1 (i)) and hence $m^*(\varnothing) = 0$.

(ii) Let $G = \bigcup_{k=1}^{\infty} \{ x : \frac{3}{2^{k+1}} < x < \frac{1}{2^{k-1}} \}$.

Then G is a bounded open subset of $(0,1)$.

We have $m(G) = \frac{1}{2}$ (Example 2.1.3) .

Therefore $m^*(G) = m(G)$ (Theorem 2.2.2)

$$= \frac{1}{2}.$$

Theorem 2.2.3

Let $E_1, E_2 \subset [a, b]$. *If* $E_1 \subset E_2$, *then*

$$m^*(E_1) \le m^*(E_2).$$

Proof

Let

$$S = \{ m(G) : G \text{ is open and } E_1 \subset G \},$$

and

$$T = \{ m(G) : G \text{ is open and } E_2 \subset G \}.$$

Let $m(G) \in T$. Then G is open and $G \supset E_2$.

Since $E_1 \subset E_2$, it follows that $G \supset E_2 \supset E_1$ and so $G \supset E_1$.

Hence $m(G) \in S$. Therefore $T \subset S$ which implies $\inf(S) \le \inf(T)$.

Thus $m^*(E_1) \le m^*(E_2)$.

Theorem 2.2.4

Let $E \subset [a, b]$ *and* $a \in \mathbb{R}$. *Then*

$$m^*(E + a) = m^*(E).$$

Proof

Let $\in > 0$. There exists an open set G containing E such that

$$m(G) < m^*(E) + \in.$$

Let $a \in \mathbb{R}$. Then $E + a \subset G + a$.

So

$$\begin{aligned} m^*(E + a) &\leq m^*(G + a) \quad \text{(Theorem 2.2.3)} \\ &= m(G + a) \quad \text{(Theorem 2.2.2)} \\ &= m(G) \quad \text{(Theorem 2.1.9)} \\ &< m^*(E) + \in. \end{aligned}$$

Since $\in$ is an arbitrary positive number, so

$$m^*(E + a) \leq m^*(E) \rightarrow \text{(i)}$$

Replacing E by $E + a$ and a by $-a$ in (i), we get

$$m^*((E + a) - a) \leq m^*(E + a).$$

Therefore

$$m^*(E) \leq m^*(E + a) \rightarrow \text{(ii)}$$

It follows from (i) and (ii) that

$$m^*(E + a) = m^*(E).$$

Propostion 2.2.5

Let $E_1, E_2 \subset [a, b]$. Then

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) < m^*(E_1) + m^*(E_2). .$$

Proof

Let $\in > 0$. There exists an open set G and $E_1 \subset G$ such that

$$m(G) < m^*(E_1) + \frac{\in}{2}.$$

Also, there exists an open set H and $E_2 \subset H$ such that

$$m(H) < m^*(E_2) + \frac{\in}{2}.$$

Then $E_1 \cup E_2 \subseteq G \cup H$ and $E_1 \cap E_2 \subseteq G \cap H$.

We have $G \cap H$ and $G \cup H$ are open.

Therefore

$$m(G) + m(H) < m^*(E_1) + m^*(E_2) + \in .$$

We have

$$m(G) + m(H) = m(G \cup H) + m(G \cap H) \quad (\text{Theorem 2.1.6}).$$

So

$$m(G \cup H) + m(G \cap H) < m^*(E_1) + m^*(E_2) + \in ,$$

and hence

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) < m^*(E_1) + m^*(E_2) + \in .$$

Since $\in$ is an arbitrary positive number, so

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) < m^*(E_1) + m^*(E_2).$$

Theorem 2.2.6

Let $E_1, E_2, \ldots, E_n$ be bounded sets. Then

$$m^*\Big(\bigcup_{k=1}^{n} E_k\Big) \leq \sum_{k=1}^{n} m^*(E_k).$$

Proof

The proof is by induction on n.

Theorem 2.2.7

Let $E_1, E_2, \ldots$ and $\bigcup_{n=1}^{\infty} E_n$ be bounded sets. Then

$$m^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leq \sum_{n=1}^{\infty} m^*(E_n).$$

Proof

Let $\in > 0$. Then for each E_n $(n = 1, 2, 3, \ldots)$, there exists an open set G_n and $E_n \subset G_n$ such that

$$m(G_n) < m^*(E_n) + \frac{\in}{2^n}.$$

We have $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} G_n$ and $\bigcup_{n=1}^{\infty} G_n$ is open.

Then
$$\begin{aligned} m^*\left(\bigcup_{n=1}^{\infty} E_n\right) &\le m\left(\bigcup_{n=1}^{\infty} G_n\right) \\ &\le \sum_{n=1}^{\infty} m(G_n) \quad (\text{Theorem } 2.1.7) \\ &< \sum_{n=1}^{\infty} \left(m^*(E_n) + \frac{\in}{2^n} \right) \\ &= \sum_{n=1}^{\infty} m^*(E_n) + \in \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} m^*(E_n) + \in. \end{aligned}$$

Thus $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n) + \in$.

Since $\in$ is an arbitrary positive number, so

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n).$$

2.3 The Lebesgue interior measure

Definition 2.3.1

Let $E \subset [a, b]$. We define the *Lebesgue interior measure* or simply *interior measure* of E, denoted by $m_*(E)$ by :

$$m_*(E) = (b - a) - m^*(E^c),$$

where $E^c = [a, b] \setminus E$.

Remarks 2.3.1

(i) Since $0 \le m^*(E^c) \le b - a$, it follows that

$$0 \le m_*(E) \le b - a.$$

Hence $m_*(E)$ is finite and exists.

(ii) It follows from the definition of an interior measure that

$$m^*(E^c) = (b-a) - m_*(E).$$

(iii) Let $E = [a, b]$. Then

$$m_*([a,b]) = m([a,b]).$$

Example 2.3.1

Let $G = \bigcup_{k=1}^{\infty} \{ x : \frac{3}{2^{k+1}} < x < \frac{1}{2^{k-1}} \}$.

Then G is a bounded open subset of $(0,1)$.

We have $m^*(G) = \frac{1}{2}$ (Example 2.2.2 (ii)).

Therefore $m_*(G) = (1-0) - m^*(G^c)$

$$= (1-0) - \frac{1}{2}$$

$$= \frac{1}{2}.$$

Thus $m_*(G) = \frac{1}{2}$.

Theorem 2.3.1

Let $E \subset [a, b]$ *and* $a \in \mathbb{R}$. *Then*

$$m_*(E+a) = m_*(E).$$

Proof

Let I be a bounded open interval containing E.

Then $E \subset I$ and $E + a \subset I + a$.

So

$$(I \setminus E) + a = (I+a) \setminus (E+a).$$

Therefore

$$m^*\big((I+a) \setminus (E+a)\big) = m^*\big((I \setminus E) + a\big)$$

$$= m^*\big(I \setminus E\big) \quad (\text{ Theorem } 2.2.4).$$

We have

$$m_*(E+a) = m(I+a) - m^*((I+a)\setminus(E+a))$$

$$= m(I) - m^*((I+a)\setminus(E+a)) \text{ (Lemma 2. 1.8)}$$

$$= m(I) - m^*(I\setminus E)$$

$$= m_*(E).$$

Theorem 2.3.2

Let $E_1, E_2 \subset [a,b]$. *If* $E_1 \subset E_2$, *then*

$$m_*(E_1) \le m_*(E_2).$$

Proof

Let $E_1, E_2 \subset [a,b]$. Then

$$m_*(E_1) = (b-a) - m^*(E_1^c)$$

$$m_*(E_2) = (b-a) - m^*(E_2^c).$$

Let $E_1 \subset E_2$. Then $E_2^c \subset E_1^c$. So

$$m^*(E_2^c) \le m^*(E_1^c) \text{ (Theorem 2.2.3)},$$

and hence

$$-m^*(E_1^c) \le -m^*(E_2^c).$$

It follows that

$$(b-a) - m^*(E_1^c) \le (b-a) - m^*(E_2^c).$$

Thus

$$m_*(E_1) \le m_*(E_2).$$

Proposition 2.3.3

Let $E_1, E_2 \subset [a,b]$. *Then*

$$m_*(E_1) + m_*(E_2) \le m_*(E_1 \cup E_2) + m_*(E_1 \cap E_2).$$

Proof

Let $E_1, E_2 \subset [a, b]$. Then

$$m_*(E_1) = (b - a) - m^*(E_1^c)$$

$$m_*(E_2) = (b - a) - m^*(E_2^c),$$

and

$$m_*(E_1 \cup E_2) = (b - a) - m^*((E_1 \cup E_2)^c).$$

We have

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) < m^*(E_1) + m^*(E_2)$$

$$(\text{ Proposition } 2.2.5) \rightarrow (1)$$

Replacing E_1, E_2 by E_1^c, E_2^c respectively and $E_1 \cup E_2$ by $(E_1 \cup E_2)^c$ and $E_1 \cap E_2$ by $(E_1 \cap E_2)^c$ in (1), we obtain

$$m^*((E_1 \cup E_2)^c) + m^*((E_1 \cap E_2)^c) \leq m^*(E_1^c) + m^*(E_2^c).$$

It follows that

$$(b - a) - m_*(E_1 \cup E_2) + (b - a) - m_*(E_1 \cap E_2) \leq (b - a) - m_*(E_1)$$

$$+ (b - a) - m_*(E_2)$$

and so

$$- m_*((E_1 \cup E_2)) - m_*(E_1 \cap E_2) \leq - m_*(E_1) - m_*(E_2).$$

Hence

$$m_*(E_1) + m_*(E_2) \leq m_*(E_1 \cup E_2) + m_*(E_1 \cap E_2).$$

Theorem 2.3.4 [7]

Let $E \subset [a, b]$. *Then*

$$m_*(E) = \sup\{ m(F) : F \text{ is closed and } F \subset E \}.$$

Theorem 2.3.5

If F *is a closed set, then* $m_*(F) = m(F)$.

Proof

Let F be a closed set . Then

$$m(F) \leq m_*(F) \quad \to (\text{i})$$

Let H be closed and $H \subset F$. Then

$$m(H) \leq m(F) \quad (\text{Lemma 2.1.11}).$$

Taking supremum of both sides over $H \subset F$. Then we have

$$\sup\{ m(H) : H \text{ is closed and } H \subset F \} \leq m(F).$$

Thus

$$m_*(F) \leq m(F) \quad \to (\text{ii})$$

It follows from (i) and (ii) that

$$m_*(F) = m(F).$$

Examples 2.3.2

(i) $m_*(\varnothing) = m(\varnothing)$.

Since $m(\varnothing) = 0$, so $m_*(\varnothing) = 0$.

(ii) $m_*(\{a\}) = m(\{a\})$.

Since $m(\{a\}) = 0$, so $m_*(\{a\}) = 0$.

Theorem 2.3.6

Let $E \subset [a, b]$. *Then*

$$m_*(E) \leq m^*(E).$$

Proof

Let G be an open set containing E and let F be a closed subset of E .

We have $F \subset E \subset G$. Then

$$m(F) \leq m(G) \quad (\text{Lemma 2.1.10}).$$

That is, $m(G)$ is an upper bound of the family $\{ m(F) \}_{F \subset G}$.

We have

$$m_*(E) = \sup\{ m(F) : F \text{ is closed and } F \subset E \} \ (\text{Theorem 2.3.4})$$

$$\leq \sup\{ m(F) : F \text{ is closed and } F \subset G \}$$

$$= m(G).$$

Thus

$$m_*(E) \leq m(G).$$

Taking infimum of both sides over $E \subseteq G$. Then we have

$$m_*(E) \leq \inf\{ m(G) : G \text{ is open and } E \subset G \}$$

$$= m^*(E).$$

Hence $m_*(E) \leq m^*(E)$.

Theorem 2.3.7 [7]

Let $F_1, F_2, \ldots F_n$ be pairwise disjoint bounded closed sets. Then

$$m\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} m(F_i).$$

Theorem 2.3.8

Let $E_1, E_2, \ldots E_n$ be pairwise disjoint bounded sets. Then

$$\sum_{i=1}^{n} m_*(E_i) \leq m_*\left(\bigcup_{i=1}^{n} E_i\right).$$

Proof

Let $\in > 0$. Then for each E_n $(n = 1, 2, 3, \ldots)$, there exists a closed set F_n and $F_n \subset E_n$ such that

$$m(F_n) > m_*(E_n) - \frac{\in}{2^n}.$$

Then the sets F_n are pairwise disjoint closed sets .

We have

$$\bigcup_{n=1}^{k} F_n \subseteq \bigcup_{n=1}^{k} E_n \quad \text{and} \quad \bigcup_{n=1}^{k} F_n \quad \text{is closed}.$$

So

$$m_*\left(\bigcup_{n=1}^{k} E_n\right) \geq m\left(\bigcup_{n=1}^{k} F_n\right)$$

$$= \sum_{n=1}^{k} m(F_n) \quad (\text{ Theorem } 2.3.7 \text{ })$$

$$> \sum_{n=1}^{k} \left(m_*(E_n) - \frac{\in}{2^n} \right)$$

$$= \sum_{n=1}^{k} m_*(E_n) - \in \sum_{n=1}^{k} \frac{1}{2^n}.$$

Thus

$$m_*\left(\bigcup_{n=1}^{k} E_n\right) \geq \sum_{n=1}^{k} m_*(E_n) - \in,$$

and hence

$$\sum_{n=1}^{k} m_*(E_n) \leq m_*\left(\bigcup_{n=1}^{k} E_n\right) + \in.$$

Since $\in$ is an arbitrary positive number, so we have

$$\sum_{n=1}^{k} m_*(E_n) \leq m_*\left(\bigcup_{n=1}^{k} E_n\right).$$

Chapter Three

Properties of the class of measurable sets

Our goal in this chapter is to give some properties of the class of measurable sets. We also obtain some useful characterizations of measurable sets.

Definition 3.1

Let $E \subset [a, b]$. Then E is called *measurable* if

$$m^*(E) = m_*(E),$$

and we write $m^*(E) = m_*(E) = m(E)$.

We give some examples concerning measurable sets.

Examples 3.1

(i) We have

$$m(\varnothing) = 0 \quad \big(\text{Lemma 2.1.1 (i)}\big)$$

$$m^*(\varnothing) = 0 \quad \big(\text{Example 2.2.2 (i)}\big)$$

$$m_*(\varnothing) = 0 \quad \big(\text{Examples 2.3.2 (i)}\big).$$

So

$$m(\varnothing) = m^*(\varnothing) = m_*(\varnothing) = 0.$$

Hence $\varnothing$ is measurable.

(ii) We have

$$m(\{a\}) = 0 \quad \big(\text{Lemma 2.1.1 (ii)}\big)$$

$$m^*(\{a\}) = 0 \quad \big(\text{Lemma 2.2.1}\big)$$

$$m_*(\{a\}) = 0 \quad \big(\text{Examples 2.3.2 (ii)}\big).$$

So

$$m(\{a\}) = m^*(\{a\}) = m_*(\{a\}) = 0.$$

Hence $\{a\}$ is measurable.

(iii) Let $G = \bigcup_{k=1}^{\infty} \left\{ x : \frac{3}{2^{k+1}} < x < \frac{1}{2^{k-1}} \right\}$.

We have

$$m(G) = \frac{1}{2} \quad (\text{Example } 2.1.3)$$

$$m^*(G) = \frac{1}{2} \quad (\text{Example } 2.2.2\,(\text{ii})),$$

and

$$m_*(G) = \frac{1}{2} \quad (\text{Example } 2.3.1).$$

So

$$m(G) = m^*(G) = m_*(G)$$

Thus G is a measurable set .

Remark 3.1

A subset of a measurable set may not be measurable , see , for example [7] .

Theorem 3.1

Let $E \subset [a, b]$. *Then* E *is measurable if and only if* E^c *is measurable* .

Proof

Let E be a measurable set . Then $m^*(E) = m_*(E)$.

We have

$$m_*(E^c) = (b - a) - m^*((E^c)^c)$$

$$= (b - a) - m^*(E)$$

$$= (b - a) - m_*(E)$$

$$= (b - a) - \big((b - a) - m^*(E^c)\big)$$

$$= m^*(E^c).$$

Hence E^c is measurable .

Conversely, let E^c be a measurable set. Then

$$m^*(E^c) = m_*(E^c).$$

We have

$$m_*(E) = (b - a) - m^*(E^c)$$

$$= (b-a) - m_*(E^c)$$

$$= (b-a) - \big((b-a) - m^*(E)\big)$$

$$= m^*(E).$$

Thus E is measurable.

Theorem 3.2

Let $E \subset [a,b]$ *and let* E *be a measurable set. Then*

$$m(E) + m(E^c) = b - a.$$

Proof

Let E be a measurable set. Then

$$m_*(E) = m^*(E) = m(E).$$

Since E^c is a measurable set (Theorem 3.1), it follows that

$$m_*(E^c) = m^*(E^c) = m(E^c).$$

We have

$$m_*(E) = (b-a) - m^*(E^c),$$

and hence

$$m(E) = (b-a) - m(E^c).$$

Thus

$$m(E) + m(E^c) = b - a.$$

Lemma 3.3

Let $E \subset [a,b]$. *If* $m^*(E) + m^*(E^c) \le b - a$, *then* E is *a measurable set*.

Proof

Let $m^*(E) + m^*(E^c) \le b - a$.

Then

$$m^*(E) \le b - a - m^*(E^c)$$

$$= m_*(E).$$

So $m^*(E) \le m_*(E)$. We have

$$m_*(E) \le m^*(E) \quad (\text{Theorem } 2.3.5).$$

Thus $m^*(E) = m_*(E)$.

Hence E is a measurable set.

Theorem 3.4

Let E be a measurable set and $a \in \mathbb{R}$. Then $E + a$ is measurable and

$$m(E + a) = m(E).$$

Proof

Let E be a measurable set. Then

$$m^*(E) = m_*(E) = m(E).$$

Let $E \subset [a, b]$ and $a \in \mathbb{R}$. Then

$$m^*(E + a) = m^*(E) \quad (\text{Theorem } 2.2.4),$$

and

$$m_*(E + a) = m_*(E) \quad (\text{Theorem } 2.3.1).$$

So we have

$$m^*(E + a) = m_*(E + a).$$

Thus $E + a$ is measurable and

$$m^*(E + a) = m_*(E + a) = m(E + a),$$

and hence

$$m(E + a) = m(E).$$

Theorem 3.5

Let E_1 and E_2 be disjoint bounded measurable sets. Then $E_1 \cup E_2$ is measurable and

$$m(E_1 \cup E_2) = m(E_1) + m(E_2).$$

Proof

Let $E_1, E_2 \subset [a,b]$. Let E_1 and E_2 be measurable sets.

Then

$$m_*(E_1) = m^*(E_1) = m(E_1),$$

and

$$m_*(E_2) = m^*(E_2) = m(E_2).$$

By definition of interior measures of E_1 and E_2, we have

$$m_*(E_1) = (b-a) - m^*(E_1^c)$$

$$m_*(E_2) = (b-a) - m^*(E_2^c).$$

It follows that

$$\left.\begin{aligned} m^*(E_1) &= (b-a) - m^*(E_1^c) \\ m^*(E_2) &= (b-a) - m^*(E_2^c). \end{aligned}\right\} \rightarrow (1)$$

We will show that $E_1 \cup E_2$ is measurable. That is, we show that

$$m_*(E_1 \cup E_2) = m^*(E_1 \cup E_2).$$

We know that $m_*(E_1 \cup E_2) \le m^*(E_1 \cup E_2)$ (Theorem 2.3.6).

It remains to show that

$$m^*(E_1 \cup E_2) \le m_*(E_1 \cup E_2).$$

Let $\in > 0$. Then there exist open sets $G_1 \supset E_1^c$ and $G_2 \supset E_2^c$ such that

$$\left.\begin{aligned} m(G_1) &< m^*(E_1^c) + \frac{\in}{2} \\ m(G_2) &< m^*(E_2^c) + \frac{\in}{2}. \end{aligned}\right\} \rightarrow (2)$$

We have $E_1 \cap E_2 = \varnothing$. So $E_1^c \cup E_2^c = [a,b]$.

Since there exist open sets $G_1 \supset E_1^c$ and $G_2 \supset E_2^c$, it follows that

$$E_1^c \cup E_2^c \subset G_1 \cup G_2 \,,$$

and hence

$$(a, b) \subset [a, b] \subset G_1 \cup G_2 \,,$$

and so we have

$$(a, b) \subset G_1 \cup G_2 \,.$$

So

$$m((a, b)) \le m(G_1 \cup G_2) \;\; (\text{ Theorem } 2.1.5\,(\text{i})\,)\,.$$

Therefore

$$b - a \le m(G_1 \cup G_2)\,.$$

Thus

$$-m(G_1 \cup G_2) \le -(b - a)\,.$$

Since

$$m(G_1 \cup G_2) = m(G_1) + m(G_2) - m(G_1 \cap G_2) \;\; (\text{ Theorem } 2.1.6\,)\,,$$

which implies

$$m(G_1 \cap G_2) = m(G_1) + m(G_2) - m(G_1 \cup G_2)\,.$$

So we have

$$m(G_1 \cap G_2) \le m(G_1) + m(G_2) - (b - a)\,.$$

Since $E_1^c \cap E_2^c \subset G_1 \cap G_2$, so we have

$$\begin{aligned} m^*((E_1 \cup E_2)^c) &= m^*(E_1^c \cap E_2^c) \\ &\le m(G_1 \cap G_2) \;\; (\text{ Definition of } m^*\,) \\ &\le m(G_1) + m(G_2) - (b - a)\,. \end{aligned}$$

It follows from (2) that

$$\begin{aligned} m^*((E_1 \cup E_2)^c) &\le \left(m^*(E_1^c) + \frac{\in}{2}\right) + \left(m^*(E_2^c) + \frac{\in}{2}\right) - (b - a) \\ &= m^*(E_1^c) + m^*(E_2^c) - (b - a) + \in. \end{aligned}$$

Since $\in$ is an arbitrary positive number, so

$$m^*((E_1 \cup E_2)^c) \le m^*(E_1^c) + m^*(E_2^c) - (b-a),$$

or

$$(b-a) - m^*(E_1^c) - m^*(E_2^c) \le -m^*((E_1 \cup E_2)^c) \to (3)$$

We have

$$m^*(E_1 \cup E_2) \le m^*(E_1) + m^*(E_2) \text{ (Propostion 2.2.5).}$$

It follows from (1) that

$$m^*(E_1 \cup E_2) \le (b-a) - m^*(E_1^c) + (b-a) - m^*(E_2^c).$$

$$= (b-a) + \big((b-a) - m^*(E_1^c) - m^*(E_2^c)\big).$$

It follows from (3) that

$$m^*(E_1 \cup E_2) \le (b-a) - m^*((E_1 \cup E_2)^c)$$

$$= m_*(E_1 \cup E_2).$$

Thus

$$m^*(E_1 \cup E_2) \le m_*(E_1 \cup E_2),$$

and hence

$$m^*(E_1 \cup E_2) = m_*(E_1 \cup E_2).$$

Thus $E_1 \cup E_2$ is measurable .

We have

$$m(E_1 \cup E_2) = m^*(E_1 \cup E_2)$$

$$\le m^*(E_1) + m^*(E_2)$$

$$= m(E_1) + m(E_2).$$

So

$$m(E_1 \cup E_2) \le m(E_1) + m(E_2).$$

Also , we have

$$m(E_1 \cup E_2) = m_*(E_1 \cup E_2)$$

$$\ge m_*(E_1) + m_*(E_2) \text{ (Propostion 2.3.3)}$$

$$= m(E_1) + m(E_2).$$

So

$$m(E_1 \cup E_2) \geq m(E_1) + m(E_2).$$

Therefore

$$m(E_1 \cup E_2) \leq m(E_1) + m(E_2) \leq m(E_1 \cup E_2).$$

Hence

$$m(E_1 \cup E_2) = m(E_1) + m(E_2).$$

We shall use the following remark in the next theorem .

Remark 3.2

Let $E_1, E_2, \dots, E_n$ be measurable sets . Then

$$m^*(E_1) = m_*(E_1) = m(E_1)$$

$$m^*(E_2) = m_*(E_2) = m(E_2),$$

and so we have

$$m^*(E_n) = m_*(E_n) = m(E_n).$$

Then

$$\sum_{i=1}^{n} m^*(E_i) = m^*(E_1) + m^*(E_2) + \dots + m^*(E_n)$$

$$= m(E_1) + m(E_2) + \dots + m(E_n)$$

$$= \sum_{i=1}^{n} m(E_i).$$

Also, we obtain $\sum_{i=1}^{n} m_*(E_i) = \sum_{i=1}^{n} m(E_i)$.

The following Theorem is a generalization of Theorem 3.5 .

Theorem 3.6

Let $E_1, E_2, \dots, E_n$ be disjoint bounded measurable sets . Then $\bigcup_{i=1}^{n} E_i$ is measurable and

$$m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Proof

It follows from induction on n that $\bigcup_{i=1}^{n} E_i$ is measurable .

That is, $m^*\left(\bigcup_{i=1}^{n} E_i\right) = m_*\left(\bigcup_{i=1}^{n} E_i\right) = m\left(\bigcup_{i=1}^{n} E_i\right)$.

We have

$$\sum_{i=1}^{n} m^*(E_i) = \sum_{i=1}^{n} m_*(E_i) = \sum_{i=1}^{n} m(E_i).$$

Since

$$m^*\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} m^*(E_i) \quad (\text{ Theorem 2.2.6 }),$$

and

$$\sum_{i=1}^{n} m_*(E_i) \le m_*\left(\bigcup_{i=1}^{n} E_i\right) \quad (\text{ Theorem 2.3.8 }).$$

It follows that

$$\sum_{i=1}^{n} m(E_i) \le m\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} m(E_i).$$

Thus

$$m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i).$$

Theorem 3.7

Let $E_1, E_2, \ldots$ *and* $\bigcup_{i=1}^{\infty} E_i$ *be bounded sets. Let* $E_1, E_2, \ldots$ *be disjoint bounded measurable sets. Then* $\bigcup_{i=1}^{\infty} E_i$ *is measurable and*

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

Proof

For every n, $\bigcup_{i=1}^{n} E_i$ is measurable (Theorem 3.6).

Then

$$\sum_{i=1}^{n} m^*(E_i) = \sum_{i=1}^{n} m(E_i)$$

$$= m\left(\bigcup_{i=1}^{n} E_i\right) \text{ (Theorem 3.6)}$$

$$= m_*\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\leq m_*\left(\bigcup_{i=1}^{\infty} E_i\right) \text{ (Theorem 2.3.2)}.$$

Since n is an arbitrary, so

$$\sum_{i=1}^{\infty} m^*(E_i) \leq m_*\left(\bigcup_{i=1}^{\infty} E_i\right).$$

We have

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) \text{ (Theorem 2.2.7)}.$$

It follows that

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) \leq m_*\left(\bigcup_{i=1}^{\infty} E_i\right) \rightarrow (1)$$

So

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq m_*\left(\bigcup_{i=1}^{\infty} E_i\right).$$

We have

$$m_*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \text{ (Theorem 2.3.5)}.$$

Therefore

$$m_*\left(\bigcup_{i=1}^{\infty} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Hence $\bigcup_{i=1}^{\infty} E_i$ is measurable.

Now, we put

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} E_i\right),$$

$$\sum_{i=1}^{\infty} m^*(E_i) = \sum_{i=1}^{\infty} m(E_i)$$

and

$$m_*\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} E_i\right).$$

in (1), then we get

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m(E_i) \leq m\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Thus

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i).$$

Proposition 3.8 [18]

Let E_1 and E_2 be measurable sets. Then

$$m^*(E_1 \cap E_2) + m^*((E_1 \cap E_2)^c) \leq b - a.$$

Corollary 3.9

Let E_1 and E_2 be measurable sets. Then $E_1 \cap E_2$ is measurable.

Proof

Let E_1 and E_2 be measurable sets. Then

$$m^*(E_1 \cap E_2) + m^*((E_1 \cap E_2)^c) \leq b - a \quad \text{(Proposition 3.8)}$$

It follows from Lemma 3.3 that $E_1 \cap E_2$ is measurable .

Corollary 3.10

Let $E_1, E_2, \ldots E_n$ be measurable sets . Then $\bigcap_{i=1}^{n} E_i$ is measurable .

Proof

Let E_i be measurable . Then E_i^c is measurable (Theorem 3.1) .

So $\bigcup_{i=1}^{n} E_i^c$ is measurable (Theorem 3. 6) and hence $\left(\bigcup_{i=1}^{n} E_i^c\right)^c$ is measurable . We have

$$\left(\bigcup_{i=1}^{n} E_i^c\right)^c = \bigcap_{i=1}^{n} E_i .$$

Hence $\bigcap_{i=1}^{n} E_i$ is measurable .

Corollary 3.11

Let $E_1, E_2, \ldots$ *be measurable sets. Then* $\bigcap_{i=1}^{\infty} E_i$ *is measurable.*

Proof

Similar to the proof of Corollary 3. 10.

Corollary 3.12

Let E_1 *and* E_2 *be measurable sets and* $E_1 \subset E_2$. *Then*

$$m(E_1) \le m(E_2).$$

Proof

Let E_1 and E_2 be measurable sets and $E_1 \subset E_2$.

We have

$$E_2 = E_1 \cup (E_2 \setminus E_1).$$

So

$$m(E_2) = m\big(E_1 \cup (E_2 \setminus E_1)\big)$$

$$= m(E_1) + m(E_2 \setminus E_1) \quad (\text{Theorem } 2.1.2).$$

Since $m(E_1)$, $m(E_2)$, $m(E_2 \setminus E_1)$ are positive, so we have

$$m(E_2) \ge m(E_1),$$

or

$$m(E_1) \le m(E_2).$$

Corollary 3.13

Let E_1 *and* E_2 *be measurable sets and* $E_1 \subset E_2$. *Then* $E_2 \setminus E_1$ *is measurable and*

$$m(E_2 \setminus E_1) = m(E_2) - m(E_1).$$

Proof

Let E_1 and E_2 be measurable sets and $E_1 \subset E_2$.

We have

$$E_2 \setminus E_1 = E_2 \cap E_1^c.$$

Since E_1 is measurable, so E_1^c is measurable.

Thus $E_2 \cap E_1^c$ is measurable (Corollary 3.9).

Hence $E_2 \setminus E_1$ is measurable .

We have

$$m(E_2) = m\big((E_2 \setminus E_1) \cup E_1\big)$$

$$= m(E_2 \setminus E_1) + m(E_1).$$

Hence

$$m(E_2 \setminus E_1) = m(E_2) - m(E_1).$$

Corollary 3.14

Let E_1 and E_2 be measurable sets. Then $E_2 \setminus E_1$ and $E_1 \setminus E_2$ are measurable .

Proof

Let E_1 and E_2 be measurable sets .

We have

$$E_2 \setminus E_1 = E_2 \setminus (E_1 \cap E_2).$$

Since E_1 and E_2 are measurable , so $E_1 \cap E_2$ is measurable (Corollary 3.9) .

Thus $E_2 \setminus (E_1 \cap E_2)$ is measurable (Corollary 3.13) .

Hence $E_2 \setminus E_1$ is measurable .

In the same way, we can prove that $E_1 \setminus E_2$ is measurable .

Theorem 3.15

If $m^*(E) = 0$, then E is a measurable set .

Proof

Let $m^*(E) = 0$.

Since $m_*(E) \leq m^*(E)$ (Theorem 2.3.6) , so

$$0 \leq m_*(E) \leq m^*(E) = 0.$$

Therefore

$$0 \leq m_*(E) \leq 0.$$

Therefore $m_*(E) = 0$. It follows that

$$m_*(E) = m^*(E) = 0.$$

Hence E is measurable.

We state and prove the next two lemmas.

Lemma 3.16

If $m^*(E) = 0$ *and* $A \subset E$*, then* $E - A$ *is measurable.*

Proof

Since $E - A \subset E$, so we have

$$m^*(E - A) \le m^*(E) \quad \text{(Theorem 2.2.3)}.$$

Let $m^*(E) = 0$. Then

$$m^*(E - A) \le 0,$$

and so we have

$$0 \le m^*(E - A) \le 0.$$

Thus

$$m^*(E - A) = 0.$$

It follows from Theorem 3.15 that $E - A$ is measurable.

Lemma 3.17

Let E *be a measurable set and* $A \subset E$*. If* $m^*(E - A) = 0$*, then* A *is measurable.*

Proof

Let E be a measurable set and $A \subset E$.

Let $m^*(E - A) = 0$. Then $E - A$ is measurable (Lemma 3.16).

Then $(E - A)^c$ is measurable (Theorem 3.1).

We have

$$A = E \cap (E - A)^c.$$

Thus A is measurable (Corollary 3.9).

Theorem 3.18 [18]

A bounded interval of $\mathbb{R}$ *is a measurable set.*

Theorem 3.19

(i) *Every bounded open set is a measurable set.*

(ii) *Every bounded closed set is a measurable set.*

Proof

(i) Let G be a bounded open set. Then

$$G = \bigcup_{i=1}^{\infty} I_i ,$$

where I_i are pairwise disjoint open intervals.

Then I_i is measurable (Theorem 3.18) and $\bigcup_{i=1}^{\infty} I_i$ is measurable (Theorem 3.7) . Hence G is measurable .

(ii) Let F be a bounded closed set. Then F^c is open and it follows from (i) that F^c is measurable . So F is measurable .

Examples 3.2

(i) Since (a, b) is a bounded open set, it follows that (a, b) is measurable (Theorem 3.19 (i)) .

(ii) Since $A = [1,2] \cup \{3\}$ is a bounded closed set, it follows that A is measurable (Theorem 3.19 (ii)) .

Propostion 3.20

Let E *be a measurable set . Then for each* $\in > 0$, *there exists an open set* $G \supset E$ *such that* $m(G - E) < \in$.

Proof

Let E be a measurable set and let G be an open set such that $E \subset G$. Then G is a measurable set (Theorem 3.19 (i)) .

It follows that $G - E$ is a measurable (Corollary 3.13) .

Let $\in > 0$. There exists an open set $G \supset E$ such that

$$m(G) < m^*(E) + \in.$$

Since E is a measurable set, so $m^*(E) = m_*(E) = m(E)$.

Therefore

$$m(G) < m(E) + \in.$$

Let G be a bounded open set of real numbers. Then

$$G = \bigcup_{i=1}^{\infty} I_i,$$

where I_i are pairwise disjoint open intervals.

We have

$$m\left(\bigcup_{i=1}^{\infty} I_i\right) < m(E) + \in,$$

and so

$$\sum_{i=1}^{\infty} m(I_i) < m(E) + \in.$$

Hence

$$\sum_{i=1}^{\infty} m(I_i) - m(E) < \in.$$

Thus

$$\begin{aligned} m(G - E) &= m(G) - m(E) \text{ (Corollary 3.13)} \\ &= \sum_{i=1}^{\infty} m(I_i) - m(E) \\ &< \in. \end{aligned}$$

Propostion 3.21

Let E be a measurable set. Then for each $\in > 0$, there exists a closed set $F \subset E$ such that $m(E - F) < \in$.

Proof

Let E be a measurable set and let F be a closed set in E. Then F is a measurable set (Theorem 3.19 (ii)).

It follows that $E - F$ is a measurable (Corollary 3.13).

Let $\in > 0$. There exists a closed set $F \subset E$ such that

$$m(F) > m_*(E) - \in.$$

Since E is a measurable set, so $m^*(E) = m_*(E) = m(E)$.
Therefore

$$m(F) > m(E) - \in.$$

Let F be a bounded closed set. Then

$$F = \bigcup_{i=1}^{k} F_i,$$

where F_i are pairwise disjoint closed sets.
We have

$$m\left(\bigcup_{i=1}^{k} F_i\right) > m(E) - \in,$$

and so

$$\sum_{i=1}^{k} m(F_i) > m(E) - \in.$$

Hence

$$m(E) - \sum_{i=1}^{k} m(F_i) < \in.$$

Thus

$$\begin{aligned} m(E - F) &= m(E) - m(F) \quad (\text{Corollary } 3.13) \\ &= m(E) - \sum_{i=1}^{k} m(F_i) \\ &< \in. \end{aligned}$$

Theorem 3.22

Let $E_1, E_2, E_3, \ldots$ *be measurable sets such that* $E_1 \supset E_2 \supset E_3 \supset \ldots$ *and* $m(E_1) < \infty$. *Then*

$$m\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{n \to \infty} m(E_n).$$

Proof

Let $E = \bigcap_{k=1}^{\infty} E_k$. We have

$$E_1 - E = (E_1 - E_2) \cup (E_2 - E_3) \cup \dots$$

Then $E_1 - E_2, E_2 - E_3, \dots$ are disjoint measurable sets .

So we have

$$m(E_1 - E) = m(E_1 - E_2) + m(E_2 - E_3) + \dots$$

Since $E_1 \supset E, E_1 \supset E_2, E_2 \supset E_3, \dots$ it follows that

$$m(E_1) - m(E) = \sum_{k=1}^{\infty} m(E_k - E_{k+1})$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n-1} m(E_k - E_{k+1})$$

$$= \lim_{n \to \infty} \big(m(E_1) - m(E_2) + m(E_2) - m(E_3) + \dots$$

$$+ m(E_{n-1}) - m(E_n) \big)$$

$$= \lim_{n \to \infty} (m(E_1) - m(E_n))$$

$$= m(E_1) - \lim_{n \to \infty} m(E_n).$$

Since $m(E_1) < \infty$, so

$$m(E) = \lim_{n \to \infty} m(E_n).$$

Thus

$$m\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{n \to \infty} m(E_n).$$

Theorem 3.23

Let $E_1, E_2, E_3, \dots$ *be measurable sets such that* $E_1 \subset E_2 \subset E_3 \subset \dots$ *Then*

$$m\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{n \to \infty} m(E_n).$$

Proof

Let $E = \bigcup_{k=1}^{\infty} E_k$. We have

$$E = E_1 \cup (E_2 - E_1) \cup (E_3 - E_2) \cup \dots$$

Then $E_1, E_2 - E_1, E_3 - E_2, \ldots$ are disjoint measurable sets.

So we have

$$m(E) = m(E_1) + m(E_2 - E_1) + m(E_3 - E_2) + \ldots$$

Since $E_1 \subset E_2 \subset E_3 \subset \ldots$, it follows that

$$m(E) = m(E_1) + \sum_{k=1}^{\infty} m(E_{k+1} - E_k)$$

$$= m(E_1) + \lim_{n \to \infty} \sum_{k=1}^{n-1} m(E_{k+1} - E_k)$$

$$= m(E_1) + \lim_{n \to \infty} \big(m(E_2) - m(E_1) + m(E_3) - m(E_2) + \ldots$$

$$+ m(E_n) - m(E_{n-1}) \big)$$

$$= m(E_1) + \lim_{n \to \infty} \big(-m(E_1) + m(E_n) \big)$$

$$= \lim_{n \to \infty} m(E_n).$$

Hence

$$m(E) = \lim_{n \to \infty} m(E_n).$$

Thus

$$m\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{n \to \infty} m(E_n).$$

Theorem 3.24

Let $E \subset [a, b]$. Then E is measurable if and only if for each $\in > 0$, there exist open sets G_1 and G_2 such that $G_1 \supset E$, $G_2 \supset E^c$ and $m(G_1 \cap G_2) < \in$.

Proof

Let $\in > 0$. Then there exist open sets $G_1 \supset E$, $G_2 \supset E^c$ such that

$$m(G_1) < m^*(E) + \frac{\in}{2}$$

$$m(G_2) < m^*(E^c) + \frac{\in}{2}.$$

Then

$$m(G_1) + m(G_2) < m^*(E) + m^*(E^c) + \in.$$

But we have

$$m(G_1) + m(G_2) = m(G_1 \cup G_2) + m(G_1 \cap G_2) \ (\text{Theorem } 2.1.6).$$

It follows that

$$m(G_1 \cup G_2) + m(G_1 \cap G_2) < m^*(E) + m^*(E^c) + \in \ \rightarrow (1)$$

Since $G_1 \supset E$ and $G_2 \supset E^c$, it follows that

$$G_1 \cup G_2 \supset E \cup E^c = [a, b].$$

We have

$$G_1 \cup G_2 \subset [a, b].$$

Hence

$$G_1 \cup G_2 = [a, b].$$

Thus

$$m(G_1 \cup G_2) = m([a, b]) = b - a.$$

It follows from (1) that

$$b - a + m(G_1 \cap G_2) < m^*(E) + m^*(E^c) + \in \ \rightarrow (2).$$

Let E be a measurable set. Then

$$m(E) + m(E^c) = b - a \ (\text{Theorem } 3.2).$$

Then (2) becomes

$$b - a + m(G_1 \cap G_2) < b - a + \in.$$

So we have

$$m(G_1 \cap G_2) < \in.$$

Conversely, let $\in > 0$. Suppose there exist open sets G_1 and G_2 such that $G_1 \supset E$, $G_2 \supset E^c$ and $m(G_1 \cap G_2) < \in$.

Since $G_1 \supset E$, $G_2 \supset E^c$, it follows that

$$m^*(E) \le m(G_1) \quad \text{and} \quad m^*(E^c) \le m(G_2).$$

Then

$$\begin{aligned} m^*(E) + m^*(E^c) &\le m(G_1) + m(G_2) \\ &= m(G_1 \cup G_2) + m(G_1 \cap G_2) \\ &\le b - a + \in. \end{aligned}$$

Hence

$$m^*(E) \le b - a - m^*(E^c) + \in,$$

and so

$$m^*(E) \le m_*(E) + \in.$$

Since $\in$ is an arbitrary postitive number , so

$$m^*(E) \le m_*(E).$$

We have

$$m_*(E) \le m^*(E) \quad (\text{Theorem } 2.3.6).$$

Hence

$$m_*(E) = m^*(E).$$

Thus E is a measurable set .

Chapter Four

Properties of the class of μ^*- measurable sets

The main aim of this chapter is to give some difference properties of the class of μ*-measurable sets .

Let us start with the following definition .

Definition 4.1

Let F be a field of subsets of X . A function $\mu : F \rightarrow \mathbb{R}$ is called *positive* if

$$\mu(A) \geq 0 \quad \text{for all} \quad A \in F .$$

Examples 4.1

(i) Let $X = \{1, 2, 3\}$.

Let $F = \{\varnothing, X, \{1\}, \{2, 3\}\}$.

Then F is a field of subsets of X .

Let $A \in F$. Define $\mu : F \rightarrow \mathbb{R}$ by

$$\mu(A) = \text{the number of elements in } A .$$

If $A_1 = \{1\}$, then $\mu(A_1) = 1$.

If $A_2 = \{2, 3\}$, then $\mu(A_2) = 2$.

If $A_3 = X = \{1, 2, 3\}$, then $\mu(A_3) = 3$.

If $A_4 = \varnothing$, then $\mu(A_4) = 0$.

Thus μ is positive .

(ii) Let $X = [-3, 7]$.

Let F = the power set of X

$= P([-3, 7])$.

Then F is a field of all subsets of X .

Let $I \in P(X)$. Define $\mu : F \rightarrow \mathbb{R}$ by

$$\mu(I) = \text{the length of the interval } I .$$

If $I_1 = [-3, -1]$, then $\mu(I_1) = 2$.

If $I_2 = [0, 1]$, then $\mu(I_2) = 1$.

If $I_3 = [4,7]$, then $\mu(I_3) = 3$.

Thus μ is positive .

Remark 4.1

For the rest of this chapter, we assume that $0 \le \mu(A) < \infty$ for all $A \in F$.

Definition 4.2

Let F be a field and let $A, B \in F$. A function $\mu : F \to \mathbb{R}$ is called *additive* if

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

where A, B are disjoint sets .

Example 4.2

Let $X = \mathbb{R}$.

Let $F = P(\mathbb{R})$.

Then F is a field of all subsets of X .

Let m be the Lebesgue measure .

Let $A \in P(\mathbb{R})$. Define $\mu : P(\mathbb{R}) \to \mathbb{R}$ by

$$\mu(A) = \lim_{n \to \infty} \frac{m(A \cap [1,n])}{n} \quad (n \in \mathbb{N}),$$

provided that the limit exists .

Let $A, B \in P(\mathbb{R})$ with $A \cap B = \varnothing$. Then

$$\begin{aligned}
\mu(A \cup B) &= \lim_{n \to \infty} \frac{m((A \cup B) \cap [1,n])}{n} \\
&= \lim_{n \to \infty} \frac{m\big((A \cap [1,n]) \cup (B \cap [1,n])\big)}{n} \\
&= \lim_{n \to \infty} \frac{m\big(A \cap [1,n]\big) + m\big(B \cap [1,n]\big)}{n} \\
&= \lim_{n \to \infty} \frac{m\big(A \cap [1,n]\big)}{n} + \lim_{n \to \infty} \frac{m\big(B \cap [1,n]\big)}{n} \\
&= \mu(A) + \mu(B).
\end{aligned}$$

Hence μ is additive .

Lemma 4.1

Let μ *be additive on a field* F. *Then* $\mu(\varnothing)=0$.

Proof

Let $A \in F$ with $\mu(A) < \infty$. Then

$$A \cup \varnothing = A.$$

So $\mu(A) = \mu(A \cup \varnothing)$

$$= \mu(A) + \mu(\varnothing).$$

Hence $\mu(\varnothing) = 0$.

Theorem 4.2

Let μ *be additive on a field* F *and let* $A, B \in F$. *If* $A \subset B$, *then*

(i) $\mu(B \setminus A) = \mu(B) - \mu(A)$

(ii) $\mu(A) \leq \mu(B)$.

Proof

(i) Let $A \subset B$. Then

$$B = A \cup (B \setminus A).$$

So

$$\mu(B) = \mu\big(A \cup (B \setminus A)\big)$$

$$= \mu(A) + \mu(B \setminus A).$$

Hence $\mu(B \setminus A) = \mu(B) - \mu(A)$.

(ii) From (i), we have

$$\mu(B) = \mu(A) + \mu(B \setminus A).$$

Since $\mu(A) \geq 0$ and $\mu(B \setminus A) \geq 0$, it follows that

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

Thus $\mu(B) \geq \mu(A)$.

Theorem 4.3

Let μ *be additive on a field* F *and let* $A, B \in F$. *If* $A \subset B$, *then*

$$\mu(A \bigcup B) \leq \mu(A) + \mu(B).$$

Proof

Let $A \subset B$. Then

$$A \bigcup B = A \cup (B \setminus A).$$

So

$$\mu(A \cup B) = \mu\big(A \cup (B \setminus A)\big)$$

$$= \mu(A) + \mu(B \setminus A).$$

Since $B \setminus A \subset B$, so by Theorem 4.2 (ii) we obtain

$$\mu(B \setminus A) \le \mu(B),$$

and hence

$$\mu(A \cup B) \le \mu(A) + \mu(B).$$

Lemma 4.4

Let μ be additive on a field F and let $A, B \in F$. Then

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B).$$

Proof

We have

$$A \setminus B = A \setminus (A \cap B).$$

So

$$\mu(A \setminus B) = \mu(A \setminus (A \cap B))$$

$$= \mu(A) - \mu(A \cap B) \text{ (Theorem 4.2 (i)).}$$

We state and prove the next two theorems .

Theorem 4.5

Let F be a field of subsets of X and let $A, B \in F$. Let μ be additive on F. Let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

If $\mu(A \Delta B) = 0$, then $\mu(A) = \mu(B)$.

Proof

Let $A, B \in F$. Then

$$\mu(A \Delta B) = \mu\big((A \setminus B) \cup (B \setminus A)\big)$$

$$= \mu(A \setminus B) + \mu(B \setminus A)$$

$$= 0.$$

Since $\mu(A) \ge 0$ for all $A \in F$ and $\mu(A \setminus B) + \mu(B \setminus A) = 0$, it follows that

$$\mu(A \setminus B) = \mu(B \setminus A) = 0.$$

We have

$$A = (A \setminus B) \cup (A \cap B).$$

Then

$$\begin{aligned} \mu(A) &= \mu((A \setminus B) \cup (A \cap B)) \\ &= \mu(A \setminus B) + \mu(A \cap B) \\ &= 0 + \mu(A \cap B) \\ &= \mu(A \cap B). \end{aligned}$$

Similarly, we have

$$B = (B \setminus A) \cup (A \cap B).$$

Then

$$\begin{aligned} \mu(B) &= \mu\big((B \setminus A) \cup (A \cap B)\big) \\ &= \mu(B \setminus A) + \mu(B \cap A) \\ &= 0 + \mu(B \cap A) \\ &= \mu(B \cap A). \end{aligned}$$

It follows that

$$\mu(A) = \mu(B).$$

Theorem 4.6

Let F be a field of subsets of X and let $A, B \in F$. Let μ be additive on F and let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Define a relation $\sim$ by

$$A \sim B \quad \text{if} \quad \mu(A \Delta B) = 0.$$

Then $\sim$ is an equivalence relation on F.

Proof

Reflexive :

$$\begin{aligned} \mu(A \Delta A) &= \mu(\varnothing) \\ &= 0. \end{aligned}$$

Thus $A \sim A$.

Symmetric :

Let $A \sim B$. Then $\mu(A \Delta B) = 0$.

Since $A \Delta B = B \Delta A$, so

$$\mu(B \Delta A) = \mu(A \Delta B) = 0.$$

Hence $B \sim A$.

Transitive :

Let $A \sim B$. Then $\mu(A \Delta B) = 0$ and $\mu(A) = \mu(B)$ (Theorem 4.5).

Let $B \sim C$. Then $\mu(B \Delta C) = 0$ and hence $\mu(B) = \mu(C)$.

Hence $\mu(A) = \mu(B) = \mu(C)$.

As in Theorem 4.5 , we can deduce that

$$\mu(B \setminus A) = \mu(B \setminus C) = 0.$$

Since $A \Delta C = (A \setminus C) \cup (C \setminus A)$, so

$$\mu(A \Delta C) = \mu(A \setminus C) + \mu(C \setminus A).$$

For $\mu(A \setminus C)$:

$$\begin{aligned} \mu(A \setminus C) &= \mu(A) - \mu(A \cap C) \quad (\text{ Lemma } 4.4) \\ &= \mu(B) - \mu(A \cap C) \\ &\le \mu(B) - \mu(A \cap C \cap B) \\ &= \mu(B \setminus (A \cap C)) \\ &= \mu\big((B \setminus A) \cup (B \setminus C)\big) \\ &\le \mu(B \setminus A) + \mu(B \setminus C) \quad (\text{ Theorem } 4.3) \\ &= 0. \end{aligned}$$

Hence $\mu(A \setminus C) = 0$.

Now, we also have that

$$\begin{aligned} \mu(A \setminus C) &= \mu(A) - \mu(A \cap C) \\ &= \mu(C) - \mu(A \cap C) \\ &= \mu(C \setminus A) \\ &= 0. \end{aligned}$$

Thus

$$\mu(A \setminus C) = \mu(C \setminus A) = 0.$$

Therefore

$$\mu(A \setminus C) + \mu(C \setminus A) = \mu((A \setminus C) \cup (C \setminus A))$$

$$= \mu(A \Delta C)$$

$$= 0.$$

Thus $A \sim C$.

Hence $\sim$ is an equivalence relation on F.

Definition 4.3

Let F be a $\sigma-$field of subsets X. A function $\mu : F \rightarrow$ □ is called an *outer measure* on F if

(i) $\mu(\emptyset) = 0$

(ii) If $A, B \in F$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$

(iii) If $A_n \in F$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

Example 4.3

Let $X = \{1, 2\}$.

Let $F = \{\emptyset, X\}$.

Then F is a $\sigma-$field of subsets of X.

Let $A \subset X$. Define $v_\alpha : F \rightarrow [0,1]$ by

$$v_\alpha(A) = \alpha \chi_A(1) + (1-\alpha)\chi_A(2) \quad (0 < \alpha < 1),$$

where χ_A is the characteristic function of A.

Then

$$v_\alpha(\emptyset) = \alpha \chi_\emptyset(1) + (1-\alpha)\chi_\emptyset(2)$$

$$= \alpha(0) + (1-\alpha)(0)$$

$$= 0.$$

Let $A \subseteq B \subset X$. Then

$$v_\alpha(A) = \alpha \chi_A(1) + (1-\alpha)\chi_A(2)$$

$$\leq \alpha \chi_B(1) + (1-\alpha)\chi_B(2)$$

$$= v_\alpha(B).$$

Let $A_n \in F$. Then

$$\nu_\alpha\left(\bigcup_{n=1}^{\infty} A_n\right) = \alpha\, \chi_{\bigcup_{n=1}^{\infty} A_n}(1) + (1-\alpha)\, \chi_{\bigcup_{n=1}^{\infty} A_n}(2)$$

$$= \sum_{n=1}^{\infty} \alpha\, \chi_{A_n}(1) + \sum_{n=1}^{\infty} (1-\alpha)\, \chi_{A_n}(2)$$

$$= \sum_{n=1}^{\infty} \left(\alpha\, \chi_{A_n}(1) + (1-\alpha)\, \chi_{A_n}(2)\right)$$

$$= \sum_{n=1}^{\infty} \nu_\alpha(A_n).$$

Thus ν_α is an outer measure .

Theorem 4.7

Let F be a $\sigma-$field and let $A, B \in F$. Let $\mu : F \to \mathbb{R}$ be an outer measure . Let $u(A) = \mu(A \cap B)$. Then u is an outer measure on F.

Proof

(i) $u(\varnothing) = \mu(\varnothing \cap B)$

$= \mu(\varnothing)$

$= 0$.

(ii) Let $A_1, A_2 \in F$ with $A_1 \subseteq A_2$. Then

$$A_1 \cap B \subseteq A_2 \cap B \ .$$

So $\mu(A_1 \cap B) \leq \mu(A_2 \cap B)$, and hence

$$u(A_1) \leq u(A_2).$$

(iii) Let $A_n \in F$. Then

$$u\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n \cap B\right)$$

$$= \mu\left(\bigcup_{n=1}^{\infty} (A_n \cap B)\right)$$

$$\leq \sum_{n=1}^{\infty} \mu(A_n \cap B)$$

$$= \sum_{n=1}^{\infty} u(A_n).$$

Thus u is an outer measure on F.

Lemma 4.8 [2]

Let F *be a* $\sigma-$*field and let* $E \in F$. *Let* $\mu^*: F \to$ *be an outer measure and let* $x \in$. *Then*

$$\mu^*(E + x) = \mu^*(E).$$

Definition 4.4

Let μ^* be an outer measure on X. A set $F \subset X$ is called *measurable with respect to* μ* or μ*-*measurable* if for every $A \subset X$, then

$$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c),$$

where A is called the *test set*.

Theorem 4.9

(i) *The universal set* X *is* μ^* - *measurable set*

(ii) *The empty set* $\varnothing$ *is* μ^* - *measurable set*.

Proof

(i) Let $A \subset X$. Then

$$\begin{aligned} \mu^*(A \cap X) + \mu^*(A \cap X^c) &= \mu^*(A \cap X) + \mu^*(A \cap \varnothing) \\ &= \mu^*(A) + \mu^*(\varnothing) \\ &= \mu^*(A) + 0 \\ &= \mu^*(A). \end{aligned}$$

Hence X is μ^* - measurable set.

(ii) Let $A \subset X$. Then

$$\begin{aligned} \mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^c) &= \mu^*(A \cap \varnothing) + \mu^*(A \cap X) \\ &= \mu^*(\varnothing) + \mu^*(A) \\ &= 0 + \mu^*(A) \\ &= \mu^*(A). \end{aligned}$$

Hence $\varnothing$ is μ^* - measurable set.

Lemma 4.10

A set $F \subset X$ *is* μ^* *- measurable if and only if for every* $A \subset X$,

$$\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c).$$

Proof

Let $A \subset X$. It is clear that if F is μ^* - measurable , then

$$\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c).$$

Conversely , let $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c)$.

Since

$$A = (A \cap F) \bigcup (A \cap F^c),$$

so we have

$$\mu^*(A) = \mu^*\big((A \cap F) \bigcup (A \cap F^c)\big)$$

$$\leq \mu^*(A \cap F) + \mu^*(A \cap F^c).$$

Thus $\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$.

Hence F is μ^* - measurable .

Lemma 4.11

Let $F \subset X$ *. Then* F *is* μ^* *- measurable set if and only if* F^c *is* μ^* *- measurable set .*

Proof

Let F be μ^* - measurable set and $A \subset X$. Then

$$\mu^*(A) = \mu^*(A \cap F) + \mu^*(A \cap F^c)$$

$$= \mu^*(A \cap F^c) + \mu^*(A \cap (F^c)^c).$$

Hence F^c is μ^* - measurable set .

Conversely, let F^c be μ^* - measurable set . Then

$$\mu^*(A) = \mu^*(A \cap F^c) + \mu^*(A \cap (F^c)^c)$$

$$= \mu^*(A \cap F^c) + \mu^*(A \cap F).$$

Hence F is μ^* - measurable set .

Theorem 4.12

Let E_1 *and* E_2 *be* μ^**-measurable sets. Then* $E_1 \cup E_2$ *is* μ^**-measurable set.*

Proof

Let E_1 be μ^*- measurable set and for any test set $A \subset X$. Then

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \rightarrow (1).$$

Now apply the definition of μ^*- measurablitity for E_2 with the test set $A \cap E_1^c$ to get

$$\mu^*(A \cap E_1^c) = \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c)$$

$$= \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1 \cup E_2)^c) \rightarrow (2)$$

It follows from (1) and (2) that

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap (E_1 \cup E_2)^c) \rightarrow (3)$$

We have

$$(A \cap E_1) \cup (A \cap E_1^c \cap E_2) = A \cap (E_1 \cup (E_1^c \cap E_2))$$

$$= A \cap ((E_1 \cup E_1^c) \cap (E_1 \cup E_2))$$

$$= A \cap (X \cap (E_1 \cup E_2))$$

$$= A \cap (E_1 \cup E_2).$$

Therefore

$$\mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) \geq \mu^*(A \cap (E_1 \cup E_2)).$$

Substituting in (3) gives

$$\mu^*(A) \geq \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)$$

It follows from Lemma 4.10 that $E_1 \cup E_2$ is μ^*- measurable set.

Corollary 4.13

Let E *and* F *be* μ^**-measurable sets. Then* $E \cap F$ *is* μ^**-measurable set.*

Proof

Let $E, F \subset X$. Then

$$E \cap F = (E^c \cup F^c)^c .$$

Since E is μ^* - measurable , so E^c is μ^* - measurable (Lemma 4.11).

Also, since F is μ^* - measurable , so F^c is μ^* - measurable .

Then $E^c \cup F^c$ is μ^* - measurable (Theorem 4.12).

It follows that $(E^c \cup F^c)^c$ is μ^* - measurable set (Lemma 4.11).

Hence $E \cap F$ is μ^* - measurable .

Corollary 4.14

Let E and F be μ^* - measurable sets . Then $E \cap F^c$ is μ^* - measurable set .

Proof

Let E, F be μ^* - measurable sets . So F^c is μ^* - measurable .

Hence $E \cap F^c$ is μ^* - measurable (Corollary 4.13) .

Corollary 4.15

Let E and F be μ^* - measurable sets and let $F \subset E$. Then $E - F$ is μ^* - measurable set .

Proof

Let E and F be μ^* - measurable sets . Then $E \cap F^c$ is μ^* - measurable (Corollary 4.14) . We have

$$E - F = E \cap F^c .$$

Hence $E - F$ is μ^* - measurable .

Theorem 4.16

Let $E_1, E_2, \ldots, E_n$ be μ^* - measurable sets . Then $\bigcup_{k=1}^{n} E_k$ is μ^* - measurable .

Proof

We use mathematical induction .

Let $n = 1$. Then for all $A \subset X$, we have

$$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c).$$

Suppose that it is true for a positive integer p $(p > 1)$. Since E_{p+1} is μ^*-measurable, it follows that

$$\mu^*(A) = \mu^*(A \cap E_{p+1}) + \mu^*(A \cap E_{p+1}^c).$$

Then

$$\mu^*(A) = \mu^*(A \cap E_{p+1}) + \mu^*(A \cap E_{p+1}^c \cap (\bigcup_{k=1}^{p} E_k)) +$$
$$\mu^*(A \cap E_{p+1}^c \cap (\bigcup_{k=1}^{p} E_k)^c).$$

Since $\bigcup_{k=1}^{p} E_k \subset E_{p+1}^c$, so we have

$$\mu^*(A) = \mu^*(A \cap E_{p+1}) + \mu^*(A \cap (\bigcup_{k=1}^{p} E_k)) +$$
$$\mu^*(A \cap E_{p+1}^c \cap (\bigcup_{k=1}^{p} E_k)^c).$$

Also, since $(\bigcup_{k=1}^{p+1} E_k)^c = E_{p+1}^c \bigcap (\bigcup_{k=1}^{p} E_k)^c$, it follows that

$$\mu^*(A) = \mu^*(A \cap E_{p+1}) + \mu^*(A \cap (\bigcup_{k=1}^{p} E_k)) + \mu^*(A \cap (\bigcup_{k=1}^{p+1} E_k)^c)$$
$$\geq \mu^*\big((A \cap E_{p+1}) \cup (A \cap (\bigcup_{k=1}^{p} E_k))\big) + \mu^*(A \cap (\bigcup_{k=1}^{p+1} E_k)^c)$$
$$= \mu^*(A \cap (\bigcup_{k=1}^{p+1} E_k)) + \mu^*(A \cap (\bigcup_{k=1}^{p+1} E_k)^c).$$

Thus $\bigcup_{k=1}^{p+1} E_k$ is μ^*- measurable.

Hence $\bigcup_{k=1}^{n} E_k$ is μ^*- measurable.

Theorem 4.17

Let $E_1, E_2, \ldots$ be μ^*- measurable sets. Then $\bigcup_{k=1}^{\infty} E_k$ is μ^*- measurable.

Proof

Let $A \subset X$. Then

$$\mu^*(A) = \mu^*(A \cap (\bigcup_{k=1}^{n} E_k)) + \mu^*(A \cap (\bigcup_{k=1}^{n} E_k)^c) \text{ (Theorem 4. 16)}$$

$$\geq \mu^*(A \cap (\bigcup_{k=1}^{n} E_k)) + \mu^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c)$$

$$\geq \sum_{k=1}^{n} \mu^*(A \cap E_k) + \mu^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c),$$

for every n. So we have

$$\mu^*(A) \geq \sum_{k=1}^{\infty} \mu^*(A \cap E_k) + \mu^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c)$$

$$\geq \mu^*(A \cap (\bigcup_{k=1}^{\infty} E_k)) + \mu^*(A \cap (\bigcup_{k=1}^{\infty} E_k)^c).$$

Hence $\bigcup_{k=1}^{\infty} E_k$ is μ^*- measurable.

Theorem 4.18

Let $\mathcal{A}$ be a family of all μ^*- measurable sets. Then $\mathcal{A}$ is a $\sigma-$field.

Proof

We can write $\mathcal{A}$ as follows:

$$\mathcal{A} = \{ F \subset X : F \text{ is } \mu^*\text{- measurable on } X \}.$$

Then

$X \in \mathcal{A}$ (Lemma 4.9 (i)).

$\varnothing \in \mathcal{A}$ (Lemma 4.9 (ii)).

Let $E \in \mathcal{A}$. Then $E^c \in \mathcal{A}$ (Lemma 4.11).

Let $E_1, E_2, \ldots \in \mathcal{A}$. Then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{A}$ (Theorem 4.17).

Thus $\mathcal{A}$ is a $\sigma-$field.

Theorem 4.19

Let $f : X \to \mathbb{R}$ be an onto function and let

$$\mathcal{A} = \{ B \subseteq \mathbb{R} : f^{-1}(B) \textit{ is } \mu^*\textit{- measurable} \}.$$

Then $\mathcal{A}$ is a $\sigma-$ field.

Proof

(i) $f^{-1}(\varnothing) = \varnothing$ is μ^*- measurable (Lemma 4.9 (ii)).

So $\varnothing \in \mathcal{A}$.

$f^{-1}(Y) = X$ is μ^* - measurable ($Y = \mathbb{R}$) (Lemma 4.9 (i)).

So $Y \in \mathcal{A}$.

(ii) Let $B \in \mathcal{A}$. Then $f^{-1}(B)$ is μ^* - measurable .

Since $f^{-1}(B^c) = (f^{-1}(B))^c$, it follows that $f^{-1}(B^c)$ is μ^* - measurable (Lemma 4.11). So $B^c \in \mathcal{A}$.

(iii) Let $B_1, B_2, \ldots \in \mathcal{A}$. Then

$$f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$$

is μ^* - measurable (Theorem 4.17). So $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$.

Hence $\mathcal{A}$ is a $\sigma -$ field .

Theorem 4.20

Let E be μ^* - measurable set and $x \in \mathbb{R}$. Then $E + x$ is μ^* - measurable set .

Proof

Let $A \subset X$. Then

$$\begin{aligned}\mu^*(A) &= \mu^*(A - x) \quad (\text{Lemma 4.8}) \\ &= \mu^*((A-x) \cap E) + \mu^*((A-x) \cap E^c) \\ &= \mu^*\big(((A-x) \cap E) + x\big) + \mu^*\big(((A-x) \cap E^c) + x\big).\end{aligned}$$

Since

$$((A - x) \cap E) + x = A \cap (E + x),$$

and

$$((A - x) \cap E^c) + x = A \cap (E + x)^c,$$

it follows that

$$\mu^*(A) = \mu^*\big(A \cap (E + x)\big) + \mu^*\big(A \cap (E + x)^c\big).$$

Hence $E + x$ is μ^* - measurable set .

Proposition 4.21

Let E be μ^* - measurable set and let $E \subset F$. Then

$$\mu^*(F \cap E^c) = \mu^*(F) - \mu^*(E).$$

Proof

Let E be μ^* - measurable set . Then for every $A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Taking $A = F$ (the test set) . Then we get

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c).$$

Since $E \subset F$, so $E \cap F = E$.

Therefore

$$\mu^*(F) = \mu^*(E) + \mu^*(F \cap E^c),$$

and so

$$\mu^*(F \cap E^c) = \mu^*(F) - \mu^*(E).$$

Theorem 4.22

Let E be μ^* - measurable set and let $F \subset X$. Then

$$\mu^*(E \cup F) + \mu^*(E \cap F) = \mu^*(E) + \mu^*(F).$$

Proof

Let E be μ^* - measurable set . Then for every $A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \rightarrow (1)$$

Taking $A = F$ (the test set) in (1) . Then we get

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c) \rightarrow (2)$$

Again , taking $A = E \cup F$ (the test set) in (1) . Then we get

$$\mu^*(E \cup F) = \mu^*((E \cup F) \cap E) + \mu^*((E \cup F) \cap E^c)$$

Since $(E \cup F) \cap E = E$ and $(E \cup F) \cap E^c = F \cap E^c$, so

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F \cap E^c) \rightarrow (3)$$

It follows from (2) and (3) that

$$\mu^*(E \cup F) + \mu^*(E \cap F) = \mu^*(E) + \mu^*(F).$$

Lemma 4.23

Let $E \subset X$ and $\mu^*(E) = 0$. Then E is μ^* - measurable set .

Proof

Let $A \subset X$. Then

$$A \setminus E = A \cap E^c .$$

Since $A \cap E^c = A \setminus E \subset A$, so $\mu^*(A \cap E^c) \le \mu^*(A)$.

Also, since $A \cap E \subset E$, so $\mu^*(A \cap E) \le \mu^*(E)$.

Therefore

$$\mu^*(A \cap E^c) + \mu^*(A \cap E) \le \mu^*(A) + \mu^*(E).$$

It follows that

$$\mu^*(A \cap E^c) + \mu^*(A \cap E) \le \mu^*(A) \quad (\text{ since } \mu^*(E) = 0).$$

By Lemma 4.10 , E is μ^* - measurable .

Lemma 4.24

Let B be μ^* - measurable . If $A \subseteq B$ and $\mu^*(B) = 0$, then A is μ^* - measurable .

Proof

Let $A \subseteq B$. Then

$$\mu^*(A) \le \mu^*(B).$$

Let $\mu^*(B) = 0$. Then

$$0 \le \mu^*(A) \le \mu^*(B) = 0.$$

So

$$\mu^*(A) = 0.$$

Hence A is μ^* - measurable (Lemma 4.23) .

Lemma 4.25

If $A \subseteq C \subseteq B$ with A, B are μ^* - measurable sets and $\mu^*(B \setminus A) = 0$, then C is μ^* - measurable .

Proof

Let $A \subseteq C \subseteq B$ and $\mu^*(B \setminus A) = 0$. Then

$$C \setminus A \subseteq B \setminus A .$$

So $C \setminus A$ is μ^* - measurable (Lemma 4.24) .

We have

$$C = (C \setminus A) \cup A .$$

Since A is μ^* - measurable and $C \setminus A$ is μ^* - measurable , so

$(C \setminus A) \cup A$ is μ^* - measurable (Theorem 4.12) .

Hence C is μ^* - measurable .

Theorem 4.26

Let F be a $\sigma-$field of subsets of X . Let $A , B \in F$ with $A \bigcap B = \varnothing$. Let A be μ^* - measurable set . Then μ^* is additive . That is,

$$\mu^* (A \bigcup B) = \mu^* (A) + \mu^* (B).$$

Proof

Let A be μ^* - measurable set . Then for every $E \subset X$, we have

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Replacing E with $E \bigcap (A \cup B)$ (the test set) , yields

$$\mu^*(E \bigcap (A \cup B)) = \mu^*(E \bigcap (A \cup B) \bigcap A) +$$

$$\mu^*(E \bigcap (A \cup B) \bigcap A^c)$$

$$= \mu^*(E \bigcap ((A \cap A) \cup (B \cap A))) +$$

$$\mu^*(E \bigcap ((A \cap A^c) \cup (B \cap A^c)))$$

$$= \mu^*(E \bigcap (A \cup \varnothing)) + \mu^*(E \bigcap (\varnothing \cup B))$$

$$= \mu^*(E \bigcap A) + \mu^*(E \bigcap B).$$

Taking $E = X$, so we have

$$\mu^*(X \bigcap (A \cup B)) = \mu^*(X \bigcap A) + \mu^*(X \bigcap B).$$

Thus $\mu^* (A \bigcup B) = \mu^* (A) + \mu^* (B).$

Chapter Five

Properties of the class of measurable functions

The class of measurable functions will play a critical role in the theory of Lebesgue integration . The concept of measurable functions is a natural outgrowth of the idea of measurable sets. Measurable functions in measure theory are analogous to continuous functions in topology. A continuous function pulls back open sets to open sets, while a measurable function pulls back measurable sets to measurable sets..

Definition 5.1

Let X be a non-empty set and let F be a $\sigma-$field of subsets of X . Then (X, F) is called a *measurable space* .

A subset E of X is said to be *measurable* if $E \in F$.

Examples 5.1

(i) Let X be a non - empty set and let $F = \{ \varnothing , X \}$.

Then F is a $\sigma-$field of subsets of X .

Thus (X, F) is a measurable space .

(ii) Let X be the set of all real numbers and let $F = P(X)$,

where $P(X)$ is a power set of X .

Then F is a $\sigma-$field of subsets of X .

Thus $(X, P(X))$ is a measurable space .

(iii) Let $X = \{1,2,3,4,5,6\}$.

Let $F = \{ \varnothing, \{1,3,5\}, \{2,4,6\}, X \}$.

Then F is a $\sigma-$field of subsets of X .

Thus (X, F) is a measurable space.

Definition 5.2

Let X be a set and let F be a σ- field of subsets of X . A function μ on F is called *measure* if

(i) $\mu(\varnothing) = 0$

(ii) If (A_n) is a disjoint sequence of sets in F , then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Example 5.2

Let $X = □$.

Let $F = P(□)$ be the family of all subsets of $□$.

Let (α_m) be a sequence of non-negative real numbers .

Let $A \in P(□)$. Define $\mu : F \to □$ by

$$\mu(\varnothing) = 0,$$

$$\mu(A) = \sum_{m \in A} \alpha_m \quad (A \neq \varnothing).$$

Let (A_n) be a disjoint sequence of sets in F. Then

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{m \in \bigcup_{n=1}^{\infty} A_n} \alpha_m \\
&= \sum_{m \in A_1 \cup A_2 \cup \ldots} \alpha_m \\
&= \sum_{m \in A_1} \alpha_m + \sum_{m \in A_2} \alpha_m + \ldots \\
&= \sum_{n=1}^{\infty} \mu(A_n).
\end{aligned}$$

Thus $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Hence μ is a measure on F .

Remark 5.1

Let X be a set and let F be a σ- field of subsets of X . If μ is a measure on F, then

$$\mu\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \mu(A_n),$$

where $A_1, A_2, \ldots, A_k$ are disjoint sets in F .

Definition 5.3

Let X be a non-empty set and F be a $\sigma-$field of subsets of X. Let μ be a measure on F. Then (X, F, μ) is called *a measure space*.

Example 5.3

Let $X = \square$.

Let $F = P(X)$ be the family of all subsets of X.

Define μ as in Example 5.2. Then $(X, P(X), \mu)$ is a measure space.

Lemma 5.1

Let (X, F, μ) be a measure space and let $\mu(A) \geq 0$ for all $A \in F$. Let $A, B \in F$. If $A \subset B$, then

$$\mu(A) \leq \mu(B).$$

Proof

Let $A \subset B$. Then

$$B = A \cup (B \setminus A).$$

So

$$\begin{aligned} \mu(B) &= \mu(A \cup (B \setminus A)) \\ &= \mu(A) + \mu(B \setminus A) \\ &\geq \mu(A). \end{aligned}$$

Thus

$$\mu(A) \leq \mu(B).$$

Lemma 5.2

Let (X, F, μ) be a measure space and let $\mu(E) \geq 0$ for all $E \in F$. Then

$$\mu(X \setminus E) = \mu(X) - \mu(E).$$

Proof

Let $E \subset X$. Then

$$X = E \cup (X \setminus E).$$

So

$$\mu(X) = \mu(E) + \mu(X \setminus E),$$

and hence

$$\mu(X \setminus E) = \mu(X) - \mu(E).$$

Most of the theory of measurable functions does not depend on the specific features of the measure space on which the functions are defined, so we consider general spaces.

Definition 5.4

Let (X, F) be a measurable space. A function $f : X \to \mathbb{R}$ is called *measurable* if for every $a \in \mathbb{R}$, then

$$\{x \in X : f(x) > a\} \in F.$$

Remark 5.2

Let (X, F) be a measurable space. It follows from Definition 5.4 that a function $f : X \to \mathbb{R}$ is measurable if and only if for all $a \in \mathbb{R}$, $f^{-1}\big((a, \infty)\big) \in F$.

Lemma 5.3

Let (X, F) be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if and only if for each real number a, then

$$\{x \in X : f(x) \leq a\} \in F.$$

Proof

Let f be a measurable function. Then for each real number a, the set

$$\{x \in X : f(x) > a\} \in F.$$

So

$$\{x \in X : f(x) > a\}^{c} \in F,$$

and hence

$$\{x \in X : f(x) \leq a\} \in F.$$

Conversely, let $\{x \in X : f(x) \leq a\} \in F$, and hence

$$\{x \in X : f(x) \leq a\}^{c} \in F.$$

Therefore

$$\{x \in X : f(x) \leq a\}^{c} = \{x \in X : f(x) > a\} \in F.$$

Hence f is a measurable function.

Remark 5.3

Let (X, F) be a measurable space. It follows from Lemma 5.3 that the function $f : X \to \mathbb{R}$ is measurable if and only if for all $a \in \mathbb{R}$,

$f^{-1}\big((-\infty, a]\big) \in F$.

Example 5.4

Let $X = \mathbb{R}$.

Let $F = \{\varnothing, (-\infty, 0], (0, \infty), \mathbb{R}\}$.

Then F is a σ-field of subsets of X.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x.$$

We have

$$\begin{aligned} f^{-1}\big((-\infty, 1]\big) &= \{x \in X : f(x) \in (-\infty, 1]\} \\ &= \{x \in X : -\infty < f(x) \le 1\} \\ &= \{x \in X : -\infty < x \le 1\} \\ &= (-\infty, 1] \notin F. \end{aligned}$$

Thus $f^{-1}\big((-\infty, 1]\big) \notin F$.

Hence f is not a measurable function on F.

Lemma 5.4

Let (X, F) be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if and only if for each real number a, then

$$\{x \in X : f(x) \ge a\} \in F.$$

Proof

Let f be a measurable function. Then for each real number a, the set

$$\{x \in X : f(x) > a\} \in F.$$

It follows that

$$\{x \in X : f(x) > a - \frac{1}{n}\} \in F \quad (n = 1, 2, 3, \ldots).$$

Thus

$$\{x \in X : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in X : f(x) > a - \frac{1}{n}\} \in F .$$

Conversely, let $\{x \in X : f(x) \geq a\} \in F$.

Then $\{x \in X : f(x) \geq a + \frac{1}{n}\} \in F$.

So

$$\{x \in X : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) \geq a + \frac{1}{n}\} \in F .$$

Hence f is a measurable function.

Remark 5.4

Let (X, F) be a measurable space. It follows from Lemma 5.4 that the function $f : X \to \mathbb{R}$ is measurable if and only if for all $a \in \mathbb{R}$, $f^{-1}([a, \infty)) \in F$.

Lemma 5.5

Let (X, F) be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if and only if for each real number a, then

$$\{x \in X : f(x) < a\} \in F .$$

Proof

Let f be a measurable function. Then for each real number a, the set

$$\{x \in X : f(x) \geq a\} \in F \quad \text{(Lemma 5.4)}.$$

We have

$$\{x \in X : f(x) < a\} = \{x \in X : f(x) \geq a\}^c \in F .$$

Conversely, let $\{x \in X : f(x) < a\} \in F$.

It follows that

$$\{x \in X : f(x) < a\}^c \in F ,$$

and so

$$\{x \in X : f(x) \geq a\} \in F .$$

Hence f is a measurable function.

Remark 5.5

Let (X, F) be a measurable space. It follows from Lemma 5.5 that the function $f : X \to \mathbb{R}$ is measurable if and only if for all $a \in \mathbb{R}$,

$f^{-1}((-\infty, a)) \in F$.

Lemma 5.6

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function and let $a \in \mathbb{R}$. Then

$$\{x \in X : f(x) = a\} \in F.$$

Proof

Let $a \in \mathbb{R}$. Then

$$\{x \in X : f(x) = a\} = \{x \in X : f(x) \le a\} \cap \{x \in X : f(x) \ge a\}.$$

Since

$$\{x \in X : f(x) \le a\} \in F \quad (\text{Lemma } 5.3),$$

and

$$\{x \in X : f(x) \ge a\} \in F \quad (\text{Lemma } 5.4),$$

so

$$\{x \in X : f(x) \le a\} \cap \{x \in X : f(x) \ge a\} \in F.$$

It follows that

$$\{x \in X : f(x) = a\} \in F.$$

Lemma 5.7

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function. Let $a, b \in \mathbb{R}$. Then

$$\{x \in X : a \le f(x) \le b\} \in F.$$

Proof

Let $a, b \in \mathbb{R}$. Then

$$\{x \in X : a \le f(x) \le b\} = \{x \in X : a \le f(x)\} \cap \{x \in X : f(x) \le b\} \in F.$$

Thus $\{x \in X : a \le f(x) \le b\} \in F$.

Lemma 5.8

Let (X, F) *be a measurable space. Let* $f : X \to \mathbb{R}$ *be a measurable function. Let* $a, b \in \mathbb{R}$. *Then* $f^{-1}\big((a, b)\big) \in F$.

Proof

Let $a, b \in \mathbb{R}$. Then

$$f^{-1}\big((a, b)\big) = f^{-1}\big((-\infty, b) \cap (a, \infty)\big)$$

$$= f^{-1}\big((-\infty, b)\big) \cap f^{-1}\big((a, \infty)\big) \in F.$$

Thus $f^{-1}\big((a, b)\big) \in F$.

Theorem 5.9

Let (X, F) *be a measurable space. Let* $f : X \to \mathbb{R}$ *be a measurable function. Then* f^n (n is a positive integer) *is measurable*.

Proof

Let $a \in \mathbb{R}$. If n is odd, then

$$\{x \in X : f^n(x) \le a\} = \{x \in X : f(x) \le a^{\frac{1}{n}}\} \in F.$$

Let $a \ge 0$. If n is even, then

$$\{x \in X : 0 \le f^n(x) \le a\} = \{x \in X : a^{-\frac{1}{n}} \le f(x) \le a^{\frac{1}{n}}\} \in F.$$

(Lemma 5.7)

Let $a < 0$. If n is even, then

$$\{x \in X : f^n(x) \le a\} = \{x \in X : f(x) \le a^{\frac{1}{n}}\}$$

$$= \varnothing \in F.$$

Thus f^n is measurable.

Lemma 5.10

Let (X, F) *be a measurable space. A constant function* $f : X \to \mathbb{R}$ *is measurable.*

Proof

Let f be a constant function. Then

$$f(x) = k \quad \text{for all } x \text{ in } X .$$

We have

$$\{ x \in X : f(x) > a \} = \begin{cases} X & \text{if } a < k \\ \varnothing & \text{if } a \geq k . \end{cases}$$

It follows that

$$\{ x \in X : f(x) > a \} \in F .$$

Hence f is measurable .

Lemma 5.11

Let (X, F) be a measurable space . Let $f : X \to \mathbb{R}$ be a measurable function and let $\lambda \in \mathbb{R}$. Then $f + \lambda$ is measurable .

Proof

Let $a \in \mathbb{R}$. Then

$$\{ x \in X : f(x) + \lambda > a \} = \{ x \in X : f(x) > a - \lambda \}$$

$$= \{ x \in X : f(x) > a_1 \} \in F ,$$

where $a_1 = a - \lambda$.

Hence $f + \lambda$ is measurable .

Theorem 5.12

Let (X, F) be a measurable space . Let $f : X \to \mathbb{R}$ be a measurable function and let $\alpha \in \mathbb{R}$. Then αf is measurable .

Proof

Let $\alpha \in \mathbb{R}$. For $\alpha \in \mathbb{R}$, we have three cases :

Case (i) : let $\alpha = 0$. Then

$$\alpha f(x) = 0 ,$$

which is measurable (Lemma 5.10) .

Case (ii) : let $\alpha > 0$ and let $a \in \mathbb{R}$. Then

$$\{ x \in X : (\alpha f)(x) > a \} = \{ x \in X : \alpha f(x) > a \}$$

$$= \{ x \in X \ : f(x) > \frac{a}{\alpha} \}$$

$$= \{ x \in X \ : f(x) > a_1 \} \in F ,$$

where $a_1 = \frac{a}{\alpha}$.

Hence αf is measurable.

Case (iii) : let $\alpha < 0$ and let $a \in \mathbb{R}$. Then

$$\{ x \in X \ : \alpha f(x) > a \} = \{ x \in X \ : f(x) < \frac{a}{\alpha} \}$$

$$= \{ x \in X \ : f(x) < a_2 \} \in F \ (\text{ Lemma 5.5 }),$$

where $a_2 = \frac{a}{\alpha}$.

Hence αf is measurable.

Proposition 5.13

Let (X, F) be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then for every $a \in \mathbb{R}$, the set

$$\{ x \in X \ : f(x) < g(x) + a \} \in F .$$

Proof

Let $a \in \mathbb{R}$. Then

$$\{ x \in X \ : f(x) < g(x) + a \} = \{ x \in X : \exists \ \mathrm{r} \in \mathbb{Q} , f(x) < \mathrm{r} < g(x) + a \}$$

$$= \bigcup_{r \in \mathbb{Q}} \{ x \in X \ : f(x) < \mathrm{r} < g(x) + a \}$$

$$= \bigcup_{r \in \mathbb{Q}} \big(\{ x \in X \ : f(x) < \mathrm{r} \} \cap \{ x \in X \ : g(x) > \mathrm{r} - a \} \big) \in F .$$

Thus

$$\{ x \in X \ : f(x) < g(x) + a \} \in F .$$

Theorem 5.14

Let (X, F) be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then $f + g$ is measurable.

Proof

Let g be a measurable function. Then $-g$ is measurable function (Theorem 5.12, $\alpha = -1$). Let $a \in \mathbb{R}$. Then

$\{x \in X : f(x) + g(x) < a\} = \{x \in X : f(x) < -g(x) + a\} \in F$

(Proposition 5.13).

Hence $f + g$ is measurable.

The next theorem is a generalization of Theorem 5.14.

Theorem 5.15

Let (X, F) be a measurable space. Let $n \in \mathbb{N}$ and let $f_1, f_2, \ldots, f_n$ be measurable functions. Then $f_1 + f_2 + \ldots + f_n$ is measurable.

Proof

We use mathematical induction.

Let $n = 1$. Then f_1 is measurable.

We assume it is true for $n = k$. That is,

$$f_1 + f_2 + \ldots + f_k$$

is measurable.

Let $n = k + 1$. We have

$$f_1 + f_2 + \ldots + f_{k+1} = (f_1 + f_2 + \ldots + f_k) + f_{k+1},$$

which is measurable (Theorem 5.14).

Hence $f_1 + f_2 + \ldots + f_n$ is measurable.

Theorem 5.16

Let (X, F) be a measurable space. Let $n \in \mathbb{N}$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be real constants. Let $f_1, f_2, \ldots, f_n$ be measurable functions. Then

$$\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$$

is measurable.

Proof

Let f_1 be a measurable function. Then $\lambda_1 f_1$ is measurable (Theorem 5.12).

Let f_2 be a measurable function. Then $\lambda_2 f_2$ is measurable.

In the same way, if f_n is measurable, then $\lambda_n f_n$ is measurable.

It follows that

$$\lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_n f_n$$

is measurable (Theorem 5.15).

Corollary 5.17

Let (X, F) be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then $f - g$ is measurable.

Proof

Let g be a measurable function. Then $(-1)\,g$ is measurable function (Theorem 5.12, $\alpha = -1$). We have

$$f - g = f + (-1)\,g .$$

Since f is measurable and $(-1)\,g$ is measurable, so $f + (-1)\,g$ is measurable (Theorem 5.14).

Hence $f - g$ is measurable.

Lemma 5.18

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function. Then $|f|$ is measurable.

Proof

Let $a \in \mathbb{R}$. Then

$$\{x \in X : |f(x)| < a\} = \{x \in X : -a < f(x) < a\}$$

$$= \{x \in X : f(x) > -a\} \cap \{x \in X : f(x) < a\} \in F .$$

Hence $|f|$ is measurable.

Theorem 5.19

Let (X, F) be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then

(i) $\{x \in X : f(x) > g(x)\} \in F$

(ii) $\{x \in X : f(x) \geq g(x)\} \in F$.

Proof

(i) $\{x \in X : f(x) > g(x)\} = \bigcup_{r \in \mathbb{Q}} \big(\{x \in X : f(x) > r\}$

$\cap \{x \in X : g(x) < r\} \big) \in F$.

Thus $\{x \in X : f(x) > g(x)\} \in F$.

(ii) $\{x \in X : f(x) \geq g(x)\} = X \setminus \{x \in X : g(x) > f(x)\} \in F$.

Thus $\{x \in X : f(x) \geq g(x)\} \in F$.

By using the idea of the measurability of functions , we state and prove the next proposition .

Proposition 5.20

Let (X, F) be a measurable space and let $f : X \to \mathbb{R}$ be a measurable function defined over E_K $(K = 1, 2, 3, ..)$ of X . Then f is a measurable function on $\bigcup_{K=1}^{\infty} E_K$.

Proof

Let $f : X \to \mathbb{R}$ be a measurable function defined over E_K $(K = 1, 2, 3, ...)$.

Then for every $a \in \mathbb{R}$,

$$\{x \in E_K : f(x) > a\} \in F .$$

We have

$$\{x \in \bigcup_{K=1}^{\infty} E_K : f(x) > a\} = \bigcup_{K=1}^{\infty} \{x \in E_K : f(x) > a\} \in F .$$

So

$$\{x \in \bigcup_{K=1}^{\infty} E_K : f(x) > a\} \in F .$$

Hence f is a measurable function on $\bigcup_{K=1}^{\infty} E_K$.

Theorem 5.21

Let (X, F) be a measurable space . Let $f : X \to \mathbb{R}$ be a measurable function and let O be an open set . Then

$$\{x \in X : f(x) \in O\} \in F .$$

Proof

Let O be an open set. Then

$$O = \bigcup_{k=1}^{\infty} I_k ,$$

where $I_k = (a_k , b_k)$ are open disjoint intervals.

Then we have

$$\{ x \in X : f(x) \in O \} = \left\{ x \in X : f(x) \in \bigcup_{k=1}^{\infty} I_k \right\}$$

$$= \bigcup_{k=1}^{\infty} \{ x \in X : f(x) \in I_k \}$$

$$= \bigcup_{k=1}^{\infty} \Big(\{ x \in X : f(x) > a_k \}$$

$$\cap \{ x \in X : f(x) < b_k \} \Big) \in F .$$

Theorem 5.22

Let (X, F) be a measurable space. Let $f, g : X \to \mathbb{R}$ be measurable functions. Then fg is measurable.

Proof

We have

$$fg = \frac{1}{4} \left((f+g)^2 - (f-g)^2 \right).$$

Since f, g are measurable functions, so $(f+g)$ is measurable function (Theorem 5.14) and hence $(f+g)^2$ is measurable function (Theorem 5.9, $n = 2$). Also, we have $(f-g)$ is a measurable function (Corollary 5.17), it follows that $(f-g)^2$ is a measurable function.

Therefore $(f+g)^2 - (f-g)^2$ is a measurable function.

Thus $fg = \frac{1}{4} \left((f+g)^2 - (f-g)^2 \right)$ is a measurable function (Theorem 5.12 $\alpha = \frac{1}{4}$). Hence fg is measurable.

Remark 5.6

Also, we can also define fg by

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right).$$

Theorem 5.23

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function. If $A \subset X$, then $f : A \to \mathbb{R}$ is measurable.

Proof

Let $f : X \to \mathbb{R}$ be a measurable function.

Then for every $a \in \mathbb{R}$, we have

$$\{x \in X : f(x) > a\} \in F.$$

Let $A \subset X$. Then $A \in F$.

We have

$$\{x \in A : f(x) > a\} = \{x \in X : f(x) > a\} \bigcap A \in F.$$

Thus

$$\{x \in A : f(x) > a\} \in F.$$

Hence $f : A \to \mathbb{R}$ is measurable.

Theorem 5.24

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function. Then $\frac{1}{f}$ $(f \neq 0)$ is measurable.

Proof

Let $a \in \mathbb{R}$. If $a > 0$, then $\frac{1}{f(x)} \le a$ if and only if $f(x) < 0$ or $\left(f(x) > 0 \text{ and } \frac{1}{a} \le f(x)\right)$.

Then we have

$$\left\{x \in X : \frac{1}{f(x)} \le a\right\} = \left(\left\{x \in X : f(x) < 0\right\} \cup\right.$$

$$\left\{ x \in X \ : \frac{1}{a} \leq f\ (x) \right\} \cap \left\{ x \in X \ : f\ (x) > 0 \right\}\Big) \in F.$$

If $a = 0$, then $\dfrac{1}{f(x)} \leq a$ if and only if $f(x) < 0$.

Then we have

$$\{ x \in X \ : \frac{1}{f(x)} \leq a \} = \{ x \in X \ : f\ (x) < 0 \} \in F\ .$$

If $a < 0$, then $\dfrac{1}{f(x)} \leq a$ if and only if $f(x) < 0$ and $\dfrac{1}{a} \leq f\ (x)$.

Then we have

$$\left\{ x \in X \ : \frac{1}{f(x)} \leq a \right\} = \Big(\{ x \in X \ : f\ (x) < 0 \} \Big) \cap$$

$$\left\{ x \in X \ : \frac{1}{a} \leq f\ (x) \right\} \in F\ .$$

Hence $\dfrac{1}{f}$ is a measurable function .

Corollary 5.25

Let (X, F) *be a measurable space . Let* $f, g : X \to \mathbb{R}$ *be measurable functions . Then* $\dfrac{f}{g}$ $(g \neq 0)$ *is measurable .*

Proof

We have

$$\frac{f}{g} = f \ \frac{1}{g} \ \ (g \neq 0).$$

Since g is measurable , so $\dfrac{1}{g}$ is measurable (Theorem 5.24) .

It follows that $f \cdot \dfrac{1}{g}$ is also measurable (Theorem 5.22) .

Thus $\dfrac{f}{g}$ is measurable .

Theorem 5.26

Let (X, F) *be a measurable space . Let* $f, g : X \to \mathbb{R}$ *be measurable functions . Then* $\max\{ f , g \}$ *and* $\min\{ f , g \}$ *are measurable .*

Proof

We have

$$\max\{f, g\} = \frac{f + g + |f - g|}{2}.$$

Since f and g are measurable, so $f + g$ is measurable (Theorem 5.14). Also, since f and g are measurable , so $f - g$ is measurable (Corollary 5.17) and so $|f - g|$ is measurable (Lemma 5.18). So we have $f + g + |f - g|$ is measurable. It follows that

$$\frac{f + g + |f - g|}{2}$$

is measurable (Theorem 5.12 , $\alpha = \frac{1}{2}$).

Hence $\max\{f, g\}$ is measurable.

We have

$$\min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

In the same way, we can prove that $\min\{f, g\}$ is measurable.

Theorem 5.27

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function. Then f^+, f^- are measurable functions.

Proof

(i) $f^+(x) = \max\{f(x), 0\}$.

Since $\max\{f(x), 0\}$ is measurable (Theorem 5.26), so f^+ is measurable.

(ii) $f^-(x) = \min\{0, -f(x)\}$.

Since $\min\{0, -f(x)\}$ is measurable (Theorem 5.26), so f^- is measurable.

Theorem 5.28

Let (X, F) *be a measurable space. Then the characteristic function* χ_E *is measurable if and only if* $E \in F$.

Proof

Let χ_E be a measurable function. Then

$$E = \{ x \in X : \chi_E(x) > 0 \} \in F .$$

Hence $E \in F$.

Conversely, let $E \in F$.

If $a \leq 0$, then $\{ x \in X : \chi_E(x) < a \} = \varnothing$ which is a measurable set.

If $a > 1$, then $\{ x \in X : \chi_E(x) < a \} = X$ which is a measurable set.

If $0 < a \leq 1$, then $\{ x \in X : \chi_E(x) < a \} = X \setminus E$ which is a measurable set.

Hence χ_E is a measurable function.

Theorem 5.29

Let (X, F) *be a measurable space. Every simple function*

$$\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

is measurable if and only if $E_1, E_2, \ldots, E_n \in F$.

Proof

It follows from Theorem 5.28 that

χ_{E_1} is a measurable function if and only if $E_1 \in F$,

and hence

$a_1 \chi_{E_1}$ is a measurable function if and only if $E_1 \in F$ (Theorem 5.12).

Aslo, we have

χ_{E_2} is a measurable function if and only if $E_2 \in F$,

and hence

$a_2 \chi_{E_2}$ is a measurable function if and only if $E_2 \in F$.

In the same way, we can obtain

$$a_n \chi_{E_n} \text{ is a measurable function if and only if } E_n \in F.$$

It follows from Theorem 5.16 that

$$a_1 \chi_{E_1} + a_2 \chi_{E_2} + \dots + a_n \chi_{E_n} \text{ is measurable if and only if}$$
$$E_1, E_2, \dots, E_n \in F.$$

Hence the simple function ϕ is measurable.

Propostion 5.30

Let (X, F) be a measurable space. A function $f : X \to \mathbb{R}$ is measurable if and only if $f^{-1}(O) \in F$ for all open sets O in $\mathbb{R}$.

Proof

Let f be a measurable function and let O be an open set in $\mathbb{R}$.

Then

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n).$$

Therefore

$$f^{-1}(O) = f^{-1}\Big(\bigcup_{n=1}^{\infty} (a_n, b_n)\Big)$$

$$= \bigcup_{n=1}^{\infty} f^{-1}\big((a_n, b_n)\big) \in F.$$

Conversely, suppose that $f^{-1}(O) \in F$ for all open sets O in $\mathbb{R}$.

Take $O = (a, \infty)$ in $\mathbb{R}$. Then

$$f^{-1}\big((a, \infty)\big) \in F.$$

Hence f is a measurable function.

Theorem 5.31 [20]

Let (X, F) be a measurable space. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is measurable.

Examples 5.5

(i) Let $f(x) = x^2 + 2x + 3$.

Then f is a continuous function. So f is measurable (Theorem 5.31).

(ii) Let $f(x) = \sin x + \cos x$.

Then f is a continuous function. So f is measurable.

(iii) Let $f(x) = x - e^{x}$.

Then f is a continuous function. So f is measurable.

(iv) Let $f(x) = \dfrac{x}{x^2 + 4}$.

Then f is a continuous function. So f is measurable.

Theorem 5.32

Let (X, F) be a measurable space. Let $f : X \to \mathbb{R}$ be a measurable function and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $g \circ f : X \to \mathbb{R}$ is measurable.

Proof

For all $a \in \mathbb{R}$, let $O_a = g^{-1}((a, \infty))$. Since $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, so O_a is an open set in $\mathbb{R}$.

We have

$$(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}((a, \infty)))$$

$$= f^{-1}(O_a) \in F \text{ (Proposition 5.30)}.$$

Hence $g \circ f$ is measurable.

Lemma 5.33

Let (X, F) be a measurable space. Let (f_n) be a sequence of measurable functions. Then $\sup_n (f_n(x))$ and $\inf_n (f_n(x))$ are measurable functions.

Proof

Let $a \in \mathbb{R}$. Then

$$\{x \in X : \sup_n (f_n(x)) > a\} = \bigcup_{n=1}^{\infty} \{x \in X : f_n(x) > a\} \in F.$$

Then $\sup_n (f_n (x))$ is measurable.

Also, we have

$$\{ x \in X : \inf_n (f_n (x)) > a \} = \bigcap_{n=1}^{\infty} \{ x \in X : f_n (x) > a \} \in F .$$

Then $\inf_n (f_n (x))$ is measurable.

Lemma 5.34

Let (X , F) be a measurable space. Let (f_n) be a sequence of measurable functions. Then $\overline{\lim} f_n (x)$ and $\underline{\lim} f_n (x)$ are measurable functions.

Proof

We have

$$\underline{\lim} f_n (x) = \sup_n (\inf_{k \geq n} (f_k (x))),$$

and

$$\overline{\lim} f_n (x) = \inf_n (\sup_{k \geq n} (f_k (x))).$$

Let

$$M_n (x) = \sup_{k \geq n} (f_k (x)),$$

and

$$m_n (x) = \inf_{k \geq n} (f_k (x)).$$

Then

$$\overline{\lim} f_n (x) = \inf_n (M_n (x))$$

and

$$\underline{\lim} f_n (x) = \sup_n (m_n (x)).$$

Thus $\overline{\lim} f_n (x)$ and $\underline{\lim} f_n (x)$ are measurable (Lemma 5.33).

Theorem 5.35

Let (X , F) be a measurable space. Let (f_n) be a sequence of measurable functions such that

$$f (x) = \lim_{n \to \infty} f_n (x) .$$

Then f is measurable.

Proof

Let $f(x) = \lim_{n \to \infty} f_n(x)$. Then

$$f(x) = \limsup_{n \to \infty} f_n(x) = \liminf_{n \to \infty} f_n(x).$$

Hence f is measurable (Lemma 5.34) .

Definition 5.5

Let (X, F, μ) be a measure space . Let (f_n) be a sequence of measurable functions . We say that (f_n) converges to a function f *almost everywhere* , denoted by $f_n \to f$ a . e if

$$\mu\left(\{x \in X : f_n(x) \not\to f(x)\}\right) = 0.$$

Definition 5.6

A measure space (X, F, μ) is called *complete* if for $A \in F$ with $\mu(A) = 0$ and $B \subset A$, then $B \in F$.

That is , any subset of a measurable set of measure zero is measurable .

Theorem 5.36

Let (X, F, μ) be a complete measure space . If $f_n \to f$ a . e , then f is a measurable function .

Proof

Let $A = \left\{x \in X : f_n(x) \not\to f(x)\right\}$.

Since $f_n \to f$ a . e , so $\mu(A) = 0$.

Let $a \in \mathbb{R}$. Then

$$\{x \in X : f(x) > a\} = (\{x \in X : f(x) > a\} \cap A) \cup$$

$$\left(\{x \in X : f(x) > a\} \cap A^c\right).$$

Since $\{x \in X : f(x) > a\} \cap A \subset A$, $\mu(A) = 0$ and (X, F, μ) is complete measure space , so we have

$$\{x \in X : f(x) > a\} \cap A \in F.$$

Also, we have

$$\{x \in X : f(x) > a\} \cap A^c = \{x \in A^c : f(x) > a\}$$

$$= \{x \in A^c : \lim_{n \to \infty} f_n(x) > a\} \in F.$$

It follows that $\{x \in X : f(x) > a\} \in F$.

Hence f is measurable.

Chapter Six

Lebesgue Integration

In this chapter, we introduce the integral of real-valued functions on an arbitrary measure space and give some of its properties. We refer to this integral as the Lebesgue integral. We carry out the definition in three ways:

- for simple functions
- for non-negative measurable functions
- for measurable functions.

6.1 The Lebesgue integral of simple functions

Definition 6.1.1

Let (X, F, m) be a measure space and $E \in F$. Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ be a simple function for real numbers a_i and measurable sets E_i.

The *Lebesgue integral* of s over E with respect to a measure m is defined by

$$\int_E s \, dm = \sum_{i=1}^{n} a_i \, m(E_i),$$

where $E_i \subseteq E$ and $0 \le m(E_i) < \infty$ $(i = 1, 2, \ldots, n)$.

Remark 6.1.1

It is clear that $\int_E s \, dm < \infty$. That is, $\int_E s \, dm$ exists.

Examples 6.1.1

(i) Let $E = [0, 2]$.

Let $s = \chi_{[\frac{1}{4^n}, \frac{6}{4^n}]}$ (the characteristic function of $[\frac{1}{4^n}, \frac{6}{4^n}]$).

Then s is a simple function.

Let m be the Lebesgue measure. We have

$$\int_{[0,2]} s \, dm = \int_{[0,2]} \chi_{[\frac{1}{4^n}, \frac{6}{4^n}]} \, dm$$

$$= \mu\left(\left[\frac{1}{4^n}, \frac{6}{4^n}\right]\right)$$

$$= \frac{6}{4^n} - \frac{1}{4^n}$$

$$= \frac{5}{4^n}.$$

(ii) Let $E = [0,7]$. Let

$$s = \chi_{[0,2]} + 2\chi_{[3,7]}.$$

Then s is a simple function . We have

$$\int_E s \, d\mu = \int_{[0,7]} (\chi_{[0,2]} + 2\chi_{[3,7]})\, d\mu$$

$$= \int_{[0,7]} \chi_{[0,2]}\, d\mu + 2\int_{[0,7]} \chi_{[3,7]}\, d\mu$$

$$= 1\mu([0,2]) + 2\mu([3,7])$$

$$= 1(2-0) + 2(7-3)$$

$$= 10.$$

Lemma 6.1.1

Let (X, F, μ) be a measure space and $E \in F$. Let $s \geq 0$ be a simple function. Then

$$\int_E s \, d\mu \geq 0.$$

Proof

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i} \geq 0$. Then $a_i \geq 0$ for all $i = 1, 2, \ldots, n$.

Since $0 \leq \mu(E_i) < \infty$ $(i = 1, 2, \ldots, n)$, it follows that

$$\int_E s \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i) \geq 0.$$

Thus

$$\int_E s \, d\mu \geq 0.$$

Proposition 6.1.2

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$ *with* $\mu(E) = 0$. *Let* s *be a simple function. Then* $\int_E s \, d\mu = 0$.

Proof

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $\mu(E) = 0$.

Since $E_i \subset E$ $(i = 1, 2, \ldots, n)$, so $\mu(E_i) \le \mu(E)$ (Lemma 5.1).

Therefore $0 \le \mu(E_i) \le \mu(E) = 0$.

It follows that $\mu(E_i) = 0$ for all $i = 1, 2, \ldots, n$.

Thus

$$\int_E s \, d\mu = \sum_{i=1}^{n} a_i \, \mu(E_i)$$

$$= 0.$$

Remark 6.1.2

Since $\mu(\emptyset) = 0$, by Proposition 6.1.2, it follows that

$$\int_{\emptyset} s \, d\mu = 0.$$

Lemma 6.1.3

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$. *Let* s *be a simple function and let* α *be a real constant. Then*

$$\int_E \alpha s \, d\mu = \alpha \int_E s \, d\mu.$$

Proof

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ be a simple function. Then

$$\int_E \alpha s \, d\mu = \int_E \alpha \left(\sum_{i=1}^{n} a_i \chi_{E_i} \right) d\mu$$

$$= \int_E \sum_{i=1}^{n} \alpha a_i \chi_{E_i} \, d\mu$$

$$= \sum_{i=1}^{n} \alpha\, a_i\, \mu(E_i)$$

$$= \alpha \sum_{i=1}^{n} a_i\, \mu(E_i)$$

$$= \alpha \int_E s\, d\mu .$$

Theorem 6.1.4

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* s, t *be simple functions. Then*

$$\int_E (s+t)\, d\mu = \int_E s\, d\mu + \int_E t\, d\mu .$$

Proof

Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $t = \sum_{i=1}^{n} b_i \chi_{E_i}$ $(a_i \neq b_i)$

be two simple functions. Then

$$\int_E (s+t)\, d\mu = \int_E \left(\sum_{i=1}^{n} a_i \chi_{E_i} + \sum_{i=1}^{n} b_i \chi_{E_i} \right) d\mu$$

$$= \int_E \sum_{i=1}^{n} (a_i + b_i)\, \chi_{E_i}\, d\mu$$

$$= \sum_{i=1}^{n} (a_i + b_i)\, \mu(E_i)$$

$$= \sum_{i=1}^{n} a_i\, \mu(E_i) + \sum_{i=1}^{n} b_i\, \mu(E_i)$$

$$= \int_E s\, d\mu + \int_E t\, d\mu .$$

Thus

$$\int_E (s+t)\, d\mu = \int_E s\, d\mu + \int_E t\, d\mu .$$

Corollary 6.1.5

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let s, t be simple functions and let α, β be real constants. Then

$$\int_E (\alpha s + \beta t)\, d\mu = \alpha \int_E s\, d\mu + \beta \int_E t\, d\mu.$$

Proof

Let s, t be simple functions and let α, β be real constants.

Then

$$\int_E (\alpha s + \beta t)\, d\mu = \int_E \alpha s\, d\mu + \int_E \beta t\, d\mu \quad (\text{Theorem } 6.1.4)$$

$$= \alpha \int_E s\, d\mu + \beta \int_E t\, d\mu \quad (\text{Lemma } 6.1.3).$$

Remarks 6.1.3

(i) Corollary 6.1.5 shows that the mapping $s \mapsto \int_E s\, d\mu$ is linear.

(ii) If $\alpha = 1$, $\beta = -1$ in Corollary 6.1.5, then

$$\int_E (s - t)\, d\mu = \int_E s\, d\mu - \int_E t\, d\mu.$$

Lemma 6.1.6

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let s, t be simple functions. If $s \le t$, then

$$\int_E s\, d\mu \le \int_E t\, d\mu.$$

Proof

Let $h = t - s$. Then $h \ge 0$ is a simple function. So

$$\int_E h\, d\mu \ge 0 \quad (\text{Lemma } 6.1.1).$$

So

$$\int_E t\, d\mu = \int_E (s + h)\, d\mu$$

$$= \int_E s \, d\mu + \int_E h \, d\mu \quad (\text{Theorem } 6.1.4)$$

$$\geq \int_E s \, d\mu.$$

Hence $\int_E s \, d\mu \leq \int_E t \, d\mu.$

Proposition 6.1.7

Let (X, F, μ) be a measure space and $E \in F$. Let s be a simple function. Then

$$\left| \int_E s \, d\mu \right| \leq \int_E |s| \, d\mu.$$

Proof

We have $-|s| \leq s \leq |s|$.

Since $-|s| \leq s$, it follows that

$$\int_E -|s| \, d\mu \leq \int_E s \, d\mu \quad (\text{Lemma } 6.1.6).$$

Also, since $s \leq |s|$, it follows that

$$\int_E s \, d\mu \leq \int_E |s| \, d\mu.$$

Therefore

$$\int_E -|s| \, d\mu \leq \int_E s \, d\mu \leq \int_E |s| \, d\mu,$$

and so

$$-\int_E |s| \, d\mu \leq \int_E s \, d\mu \leq \int_E |s| \, d\mu \quad (\text{Lemma } 6.1.3,\ a = -1).$$

Thus $\left| \int_E s \, d\mu \right| \leq \int_E |s| \, d\mu.$

Theorem 6.1.8 [9]

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* $s = \sum_{i=1}^{n} a_i \chi_{A_i}$ *be a non-negative simple function and* $A_i \in F$ $(i = 1, 2, \dots, n)$. *Then*

$$\int_E s \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E).$$

Remark 6.1.4

If $E = X$ in Theorem 6.1.8 , then

$$\int_X s \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i).$$

Proposition 6.1.9

Let (X, F, μ) *be a measure space and let* $A, B \in F$ *with* $A \cap B = \emptyset$. *Let* s *be a non-negative simple function . Then*

$$\int_{A \cup B} s \, d\mu = \int_A s \, d\mu + \int_B s \, d\mu.$$

Proof

Let $E \in F$ and $A_i \in F$ $(i = 1, 2, \dots, n)$. Then

$$\int_E s \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i \cap E) \quad (\text{Theorem 6.1.8}).$$

Therefore

$$\int_{A \cup B} s \, d\mu = \sum_{i=1}^{n} a_i \, \mu\big(A_i \cap (A \cup B)\big)$$

$$= \sum_{i=1}^{n} a_i \, \mu\big((A_i \cap A) \cup (A_i \cap B)\big)$$

$$= \sum_{i=1}^{n} a_i \big(\mu(A_i \cap A) + \mu(A_i \cap B)\big)$$

$$= \sum_{i=1}^{n} a_i \, \mu(A_i \cap A) + \sum_{i=1}^{n} a_i \, \mu(A_i \cap B)$$

$$= \int_A s \, d\mu + \int_B s \, d\mu.$$

The next theorem is a generalization of Proposition 6.1.9 .

Theorem 6.1.10

Let (X, F, μ) *be a measure space and let* $A_1, A_2, \ldots, A_m \in F$ *with* $A_i \cap A_m = \varnothing \; (i \neq m)$. *Let* s *be a non-negative simple function.*

Then

$$\int_{\bigcup_{k=1}^{m} A_k} s \, d\mu = \sum_{k=1}^{m} \int_{A_k} s \, d\mu .$$

Proof

Let $A_1, A_2, \ldots, A_m \in F$ with $A_i \cap A_m = \varnothing \; (i \neq m)$.

Then

$$\int_{\bigcup_{k=1}^{m} A_k} s \, d\mu = \sum_{i=1}^{n} a_i \, \mu\left(A_i \cap \left(\bigcup_{k=1}^{m} A_k \right) \right) \quad \text{(Theorem 6.1.8)}$$

$$= \sum_{i=1}^{n} a_i \sum_{k=1}^{m} \mu(A_i \cap A_k)$$

$$= \sum_{k=1}^{m} \sum_{i=1}^{n} a_i \, \mu(A_i \cap A_k)$$

$$= \sum_{k=1}^{m} \int_{A_k} s \, d\mu .$$

Theorem 6.1.11 [9]

Let (X, F, μ) *be a measure space and let* $A_n \in F$ *such that* $\bigcup_{n=1}^{\infty} A_n = X$. *Let* s *be a non-negative simple function. Then*

$$\lim_{n \to \infty} \int_{A_n} s \, d\mu = \int_X s \, d\mu .$$

Lemma 6.1.12

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* s *be a non-negative simple function and let*

$$f(E) = \int_E s\, d\mu .$$

Then f is a measure on F.

Proof

(i) $f(\emptyset) = \int_{\emptyset} s\, d\mu = 0$ (Remark 6.1.2) .

(ii) Let $A_i \in F$. We have

$$\int_E s\, d\mu = \sum_{i=1}^{n} a_i\, \mu(A_i \cap E),$$

and so

$$f(E) = \sum_{i=1}^{n} a_i\, \mu(A_i \cap E).$$

Let $E_1, E_2, \ldots \in F$ and $E_i \cap E_j = \emptyset \ (i \neq j)$.

Then

$$f\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{i=1}^{n} a_i\, \mu\Big(A_i \cap \Big(\bigcup_{k=1}^{\infty} E_k\Big)\Big)$$

$$= \sum_{i=1}^{n} a_i \sum_{k=1}^{\infty} \mu(A_i \cap E_k)$$

$$= \sum_{k=1}^{\infty} \sum_{i=1}^{n} a_i\, \mu(A_i \cap E_k)$$

$$= \sum_{k=1}^{\infty} f(E_k).$$

Thus

$$f\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} f(E_k).$$

Hence f is a measure on F.

\Theorem 6.1.13

Let (X, F, μ) be a measure space and $E \in F$. Let s be a non-negative simple function. Then

$$\int_X s\, d\mu = \int_E s\, d\mu + \int_{X \setminus E} s\, d\mu .$$

Proof

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let s be a non-negative simple function such that

$$\varphi(E) = \int_E s \, d\mu .$$

Then φ is a measure on $\mathcal{F}$ (Lemma 6.1.12).

Let $E \subseteq X$. Then

$$\int_E s \, d\mu + \int_{X \setminus E} s \, d\mu = \varphi(E) + \varphi(X \setminus E).$$

Since $\varphi(X \setminus E) = \varphi(X) - \varphi(E)$ (Lemma 5.2), it follows that

$$\begin{aligned} \int_E s \, d\mu + \int_{X \setminus E} s \, d\mu &= \varphi(E) + \varphi(X) - \varphi(E) \\ &= \varphi(X) \\ &= \int_X s \, d\mu . \end{aligned}$$

Corollary 6.1.14

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$ with $\mu(E) = 0$. Let s be a non-negative simple function. Then

$$\int_X s \, d\mu = \int_{X \setminus E} s \, d\mu .$$

Proof

It follows from Theorem 6.1.13 that

$$\int_X s \, d\mu = \int_E s \, d\mu + \int_{X \setminus E} s \, d\mu .$$

Since $\int_E s \, d\mu = 0$ (Proposition 6.1.2), it follows that

$$\int_X s \, d\mu = \int_{X \setminus E} s \, d\mu .$$

6.2 The Lebesgue integral of non-negative measurable functions

Definition 6.2.1

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f be a non-negative bounded measurable function on E. The *Lebesgue integral* of f over E with respect to a measure μ is defined by

$$\int_E f\, d\mu = \sup\left\{ \int_E s\, d\mu : 0 \le s(x) \le f(x) \text{ for all } x \in E, s \text{ is simple} \right\},$$

or briefly, we write

$$\int_E f\, d\mu = \sup\left\{ \int_E s\, d\mu : 0 \le s \le f, s \text{ is simple} \right\}$$

$$= \sup_{s \le f} \left(\int_E s\, d\mu \right).$$

Remark 6.2.1

It is clear that $\int_E f\, d\mu < \infty$. That is, $\int_E f\, d\mu$ exists.

Lemma 6.2.1

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$. *Let* f *be a non-negative bounded measurable function on* E. *If* $\mu(E) = 0$, *then*

$$\int_E f\, d\mu = 0.$$

Proof

Let E be a measurable set with $\mu(E) = 0$.

Let s be a simple function. Then

$$\int_E s\, d\mu = 0 \quad \text{(Proposition 6.1.2)}.$$

Therefore

$$\int_E f\, d\mu = \sup_{s \le f} \left(\int_E s\, d\mu \right)$$

$$= 0.$$

Lemma 6.2.2

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f, g be non-negative bounded measurable functions on E. If $f \le g$, then

$$\int_E f \, d\mu \le \int_E g \, d\mu.$$

Proof

Let $0 \le s \le f$ and $f \le g$. Then $s \le g$.

Since

$$\int_E g \, d\mu = \sup_{s \le g} \left(\int_E s \, d\mu \right),$$

it follows from the definition of a supremum that

$$\int_E s \, d\mu \le \int_E g \, d\mu.$$

Taking supremum over $s \le f$, we have

$$\sup_{s \le f} \left(\int_E s \, d\mu \right) \le \int_E g \, d\mu.$$

Hence

$$\int_E f \, d\mu \le \int_E g \, d\mu.$$

Lemma 6.2.3

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f be a non-negative bounded measurable function on E. Let $\alpha \ge 0$. Then

$$\int_E \alpha f \, d\mu = \alpha \int_E f \, d\mu.$$

Proof

Let f be a non-negative measurable function.

Let $0 \le s \le f$ and $\alpha > 0$. Then $0 \le \alpha s \le \alpha f$.

So αs is a simple function and αf is a non-negative measurable function.

We have

$$\int_E \alpha f \, d\mu = \sup_{\alpha s \le \alpha f} \left(\int_E \alpha s \, d\mu \right)$$

$$= \sup_{\alpha s \le \alpha f} \left(\alpha \int_E s \, d\mu \right) \quad (\text{ Lemma } 6.1.3 \text{ })$$

$$= \alpha \sup_{s \le f} \left(\int_E s \, d\mu \right)$$

$$= \alpha \int_E f \, d\mu .$$

Theorem 6.2.4

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* f, g *be non-negative bounded measurable functions on* E. *Then*

$$\int_E (f+g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu .$$

Proof

Let f, g be non - negative bounded measurable functions .

Let s, t be simple functions such that $0 \le s \le f$ and $0 \le t \le g$.

Then $s+t$ is a simple function and $f+g$ is a non - negative measurable function . So $0 \le s+t \le f+g$.

We have

$$\int_E (f+g) \, d\mu = \sup_{s+t \le f+g} \left(\int_E (s+t) \, d\mu \right).$$

It follows that

$$\int_E (f+g) \, d\mu \ge \int_E (s+t) \, d\mu$$

$$= \int_E s \, d\mu + \int_E t \, d\mu \quad (\text{ Theorem } 6.1.4 \text{ }) .$$

Taking supremum over s and t, we have

$$\int_E (f + g) \, d\mu \geq \int_E f \, d\mu + \int_E g \, d\mu \quad \rightarrow (1)$$

Let v be a simple function such that $v = t + s$. Then

$$\int_E v \, d\mu = \int_E (s + t) \, d\mu$$

$$= \int_E s \, d\mu + \int_E t \, d\mu$$

$$\leq \int_E f \, d\mu + \int_E g \, d\mu.$$

Taking supremum over v, we have

$$\int_E (f + g) \, d\mu \leq \int_E f \, d\mu + \int_E g \, d\mu \quad \rightarrow (2)$$

It follows from (1) and (2) that

$$\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu .$$

Corollary 6.2.5

Let (X , F , μ) be a measure space and $E \in F$. Let f , g be non-negative bounded measurable functions on E and let $\alpha , \beta \geq 0$.

Then

$$\int_E (\alpha f + \beta g) \, d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu .$$

Proof

The proof follows from Lemma 6.2.3 and Theorem 6.2.4 .

Remark 6.2.2

Corollary 6.2.5 shows that the mapping $f \mapsto \int_E f \, d\mu$ is linear .

We have the following deduced lemmas.

Lemma 6.2.6

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* $f, g : X \to \mathbb{R}$ *be measurable functions. Then*

$$(\mathrm{i}) \quad \int_E (f^+ + g^+)\, d\mu = \int_E f^+ d\mu + \int_E g^+ d\mu$$

$$(\mathrm{ii}) \quad \int_E (f^- + g^-)\, d\mu = \int_E f^- d\mu + \int_E g^- d\mu$$

$$(\mathrm{iii}) \quad \int_E (f^+ + g^-)\, d\mu = \int_E f^+ d\mu + \int_E g^- d\mu.$$

Proof

The proof follows from Theorem 6.2.4.

Lemma 6.2.7

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* $f : X \to \mathbb{R}$ *be measurable function and let* $\alpha \geq 0$. *Then*

$$(\mathrm{i}) \quad \int_E \alpha f^+ d\mu = \alpha \int_E f^+ d\mu$$

$$(\mathrm{ii}) \quad \int_E \alpha f^- d\mu = \alpha \int_E f^- d\mu.$$

Proof

The proof follows from Lemma 6.2.3.

Lemma 6.2.8

Let (X, F, μ) *be a measure space and* $E \in F$. *Let* f, g *be non-negative bounded measurable functions on* E. *If* $f \geq g$, *then*

$$\int_E (f - g)\, d\mu = \int_E f\, d\mu - \int_E g\, d\mu.$$

Proof

We have

$$f = (f - g) + g.$$

Then

$$\int_E f\,d\mu = \int_E \big((f-g)+g\big)\,d\mu$$

$$= \int_E (f-g)\,d\mu + \int_E g\,d\mu \quad (\text{Theorem } 6.2.4).$$

Therefore

$$\int_E (f-g)\,d\mu = \int_E f\,d\mu - \int_E g\,d\mu .$$

Lemma 6.2.9

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let $f, g : X \to \mathbb{R}$ be measurable functions such that $f^+ \geq g^+$ and $f^- \geq g^-$.

Then

$$(\text{i}) \quad \int_E (f^+ - g^+)\,d\mu = \int_E f^+\,d\mu - \int_E g^+\,d\mu$$

$$(\text{ii}) \quad \int_E (f^- - g^-)\,d\mu = \int_E f^-\,d\mu - \int_E g^-\,d\mu .$$

Proof

The proof follows from Lemma 6.2.8 .

Theorem 6.2.10 [19]

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f be a non-negative bounded measurable function on E. Then

$$\int_E f\,d\mu = \int_X f\chi_E\,d\mu .$$

Propsition 6.2.11

Let $(X, \mathcal{F}, \mu)$ be a measure space. Let f be a non-negative bounded measurable function on X. Let $A, B \in \mathcal{F}$ such that $A \subseteq B$. Then

$$\int_A f\,d\mu \leq \int_B f\,d\mu .$$

Proof

Let $A \subseteq B$. Then $\chi_A \leq \chi_B$. So $f\chi_A \leq f\chi_B$.

Therefore $f\chi_A$ and $f\chi_B$ are non-negative measurable functions. It follows that

$$\int_X f\chi_A \, d\mu \le \int_X f\chi_B \, d\mu \quad (\text{Lemma } 6.2.2).$$

Hence

$$\int_A f \, d\mu \le \int_B f \, d\mu \quad (\text{Theorem } 6.2.10).$$

Proposition 6.2.12

Let (X, F, μ) be a measure space. Let f be a non-negative bounded measurable function on X and let $\alpha \in (0, \infty)$. Then

$$\mu\{x \in X : f(x) \ge \alpha\} \le \frac{1}{\alpha}\int_X f \, d\mu.$$

Proof

Let $A = \{x \in X : f(x) \ge \alpha\}$. Then

$$\begin{aligned}\int_X f \, d\mu &\ge \int_A f \, d\mu \\ &\ge \int_A \alpha \, d\mu \\ &= \alpha \int_A d\mu \\ &= \alpha\,\mu(A).\end{aligned}$$

Thus

$$\frac{1}{\alpha}\int_X f \, d\mu \ge \mu(A),$$

and hence

$$\mu\{x \in X : f(x) \ge \alpha\} \le \frac{1}{\alpha}\int_X f \, d\mu.$$

6.3 The Lebesgue integral of measurable functions

Definition 6.3.1

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \subset \mathcal{F}$. Let f be an arbitrary bounded measurable function on E (not necessarily $f \geq 0$). Then f is called *Lebesgue integrable* on E or briefly *integrable* if

$$\int_E f^+ \, d\mu < \infty \quad \text{and} \quad \int_E f^- \, d\mu < \infty .$$

The *Lebesgue integral* of f with respect to a measure μ is defined by

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu.$$

Remark 6.3.1

We have

$$|f| = f^+ + f^- .$$

Then

$$\int_E |f| \, d\mu = \int_E (f^+ + f^-) \, d\mu$$

$$= \int_E f^+ \, d\mu + \int_E f^- \, d\mu \text{ (Theorem 6.2.4).}$$

Theorem 6.3.1

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \subset \mathcal{F}$. Let f be a bounded measurable function on E. Then f is integrable if and only if

$$\int_E |f| \, d\mu < \infty.$$

Proof

Let f be an integrable function on E. Then

$$\int_E f^+ \, d\mu < \infty \quad \text{and} \quad \int_E f^- \, d\mu < \infty .$$

We have

$$\int_E |f|\, d\mu = \int_E f^+\, d\mu + \int_E f^-\, d\mu \quad (\text{Remark } 6.3.1).$$

Thus $\int_E |f|\, d\mu < \infty$.

Conversely, let $\int_E |f|\, d\mu < \infty$.

Since $f^+ \leq |f|$, so $\int_E f^+\, d\mu \leq \int_E |f|\, d\mu$

and so $\int_E f^+\, d\mu < \infty$.

Also, since $f^- \leq |f|$, so $\int_E f^-\, d\mu \leq \int_E |f|\, d\mu$

and so $\int_E f^-\, d\mu < \infty$.

Thus f is integrable.

Lemma 6.3.2

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f be a bounded measurable function on E and let $a \in \mathbb{R}$. Then

$$\int_E a f\, d\mu = a \int_E f\, d\mu.$$

Proof

Let $a \in \mathbb{R}$. Then we have two cases.

Case (i) : let $a \geq 0$.

The Lebesgue integral of $a f$ is given by

$$\int_E a f\, d\mu = \int_E (a f)^+\, d\mu - \int_E (a f)^-\, d\mu$$

$$= \int_E a f^+\, d\mu - \int_E a f^-\, d\mu$$

$$= \alpha \int_E f^{+} \, d\mu - \alpha \int_E f^{-} \, d\mu \quad (\text{ Lemma } 6.2.7)$$

$$= \alpha \left(\int_E f^{+} \, d\mu - \int_E f^{-} \, d\mu \right)$$

$$= \alpha \int_E f \, d\mu .$$

Case (ii) : let $\alpha < 0$. Then

$$\int_E \alpha f \, d\mu = \int_E (\alpha f)^{+} \, d\mu - \int_E (\alpha f)^{-} \, d\mu$$

$$= \int_E -\alpha f^{-} \, d\mu - \int_E -\alpha f^{+} \, d\mu .$$

Since $-\alpha > 0$, it follows that

$$\int_E \alpha f \, d\mu = -\alpha \int_E f^{-} \, d\mu + \alpha \int_E f^{+} \, d\mu \ (\text{ by (i) })$$

$$= \alpha \left(\int_E f^{+} \, d\mu - \int_E f^{-} \, d\mu \right)$$

$$= \alpha \int_E f \, d\mu .$$

Theorem 6.3.3 [19]

Let (X, F, μ) be a measure space and $E \in F$. Let f be a bounded measurable function on E. Then

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu .$$

Theorem 6.3.4

Let (X, F, μ) be a measure space and $E \in F$. Let f, g be bounded measurable functions on E. Then

$$\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu .$$

Proof

We have

$$\int_E (f+g)\,d\mu = \int_X (f+g)\chi_E\,d\mu \quad (\text{Theorem } 6.3.3)$$

$$= \int_X ((f+g)\chi_E)^+\,d\mu - \int_X ((f+g)\chi_E)^-\,d\mu$$

$$= \int_X (f\chi_E + g\chi_E)^+\,d\mu - \int_X (f\chi_E + g\chi_E)^-\,d\mu$$

$$= \int_X \big((f\chi_E)^+ + (g\chi_E)^+\big)\,d\mu$$

$$- \int_X \big((f\chi_E)^- + (g\chi_E)^-\big)\,d\mu$$

$$= \int_X (f\chi_E)^+\,d\mu + \int_X (g\chi_E)^+\,d\mu - \int_X (f\chi_E)^-\,d\mu$$

$$- \int_X (g\chi_E)^-\,d\mu$$

$$= \Big(\int_X (f\chi_E)^+\,d\mu - \int_X (f\chi_E)^-\,d\mu\Big)$$

$$+ \Big(\int_X (g\chi_E)^+\,d\mu - \int_X (g\chi_E)^-\,d\mu\Big)$$

$$= \int_X f\chi_E\,d\mu + \int_X g\chi_E\,d\mu$$

$$= \int_E f\,d\mu + \int_E g\,d\mu$$

Corollary 6.3.5

Let (X, F, μ) be a measure space and $E \in F$. Let f, g be bounded measurable functions on E and let α, β be real constants. Then

$$\int_E (\alpha f + \beta g)\,d\mu = \alpha\int_E f\,d\mu + \beta\int_E g\,d\mu.$$

Proof

The proof follows from Lemma 6.3.2 and Theorem 6.3.4 .

Remarks 6.3.2

(i) Corollary 6.3.5 shows that the mapping $f \mapsto \int_E f \, d\mu$ is linear .

(ii) Let $a = 1$ and $b = -1$ in Corollary 6.3.5 . Then

$$\int_E (f - g)\, d\mu = \int_E f \, d\mu - \int_E g \, d\mu .$$

Lemma 6.3.6

Let (X, F, μ) be a measure space and $E \in F$. Let f, g be bounded measurable functions on E . If $f \le g$, then $\int_E f \, d\mu \le \int_E g \, d\mu$.

Proof

Let $f \le g$. Then $f^+ - f^- \le g^+ - g^-$.

Therefore we have

$$f^+ \le g^+ \quad \text{and so} \quad \int_E f^+ \, d\mu \le \int_E g^+ \, d\mu \ (\text{ Lemma 6.2.2 }),$$

and

$$g^- \le f^- \quad \text{and so} \quad \int_E g^- \, d\mu \le \int_E f^- \, d\mu,$$

and hence

$$-\int_E f^- \, d\mu \le -\int_E g^- \, d\mu .$$

Thus

$$\int_E g \, d\mu = \int_E g^+ \, d\mu - \int_E g^- \, d\mu$$

$$\ge \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

$$= \int_E f \, d\mu .$$

Proposition 6.3.7

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$. *Let* f *be a bounded measurable function on* E. *Then*

$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu .$$

Proof

Since $-|f| \leq f \leq |f|$, it follows that

$$-\int_E |f| \, d\mu \leq \int_E f \, d\mu \leq \int_E |f| \, d\mu \quad (\text{Lemma } 6.3.6).$$

Thus $\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu .$

Theorem 6.3.8

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E_1, E_2 \in \mathcal{F}$ *with* $E_1 \cap E_2 = \emptyset$. *Let* f *be a bounded measurable function on* E. *Then*

$$\int_{E_1 \cup E_2} f \, d\mu = \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu .$$

Proof

We have

$$\int_{E_1 \cup E_2} f \, d\mu = \int_X f \chi_{E_1 \cup E_2} \, d\mu \quad (\text{Theorem } 6.3.3)$$

$$= \int_X (f \chi_{E_1 \cup E_2})^+ \, d\mu - \int_X (f \chi_{E_1 \cup E_2})^- \, d\mu$$

$$= \int_X f^+ (\chi_{E_1} + \chi_{E_2}) \, d\mu - \int_X f^- (\chi_{E_1} + \chi_{E_2}) \, d\mu$$

$$= \int_X f^+ \chi_{E_1} \, d\mu + \int_X f^+ \chi_{E_2} \, d\mu$$

$$- \int_X f^- \chi_{E_1} \, d\mu - \int_X f^- \chi_{E_2} \, d\mu$$

$$= \left(\int_X (f \chi_{E_1})^+ \, d\mu - \int_X (f \chi_{E_1})^- \, d\mu \right) + \left(\int_X (f \chi_{E_2})^+ \, d\mu - \int_X (f \chi_{E_2})^- \, d\mu \right)$$

$$= \int_X f \chi_{E_1} \, d\mu + \int_X f \chi_{E_2} \, d\mu$$

$$= \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu$$

Theorem 6.3.9

Let (X, F, μ) be a measure space and $E_1, E_2, \ldots, E_n \subset F$ with $E_n \cap E_j = \emptyset \ (n \neq j)$. Let f be a bounded measurable function on E. Then

$$\int_{\bigcup_{k=1}^{n} E_k} f \, d\mu = \sum_{k=1}^{n} \int_{E_k} f \, d\mu .$$

Proof

Let $E_1, E_2, \ldots, E_n \subset F$ with $E_n \cap E_j = \emptyset \ (h \neq j)$.

Then

$$\int_{\bigcup_{k=1}^{n} E_k} f \, d\mu = \int_X f \chi_{\bigcup_{k=1}^{n} E_k} \, d\mu$$

$$= \int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^+ d\mu - \int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^- d\mu$$

$$= \int_X f^+ \left(\chi_{E_1} + \chi_{E_2} + \ldots + \chi_{E_n} \right) d\mu$$

$$- \int_X f^- (\chi_{E_1} + \chi_{E_2} + \dots + \chi_{E_n}) \, d\mu$$

$$= \int_X (f^+ \chi_{E_1} - f^- \chi_{E_1}) \, d\mu + \int_X (f^+ \chi_{E_2} - f^- \chi_{E_2}) \, d\mu$$

$$+ \dots + \int_X (f^+ \chi_{E_n} - f^- \chi_{E_n}) \, d\mu$$

$$= \int_X f \chi_{E_1} \, d\mu + \int_X f \chi_{E_2} \, d\mu + \dots + \int_X f \chi_{E_n} \, d\mu$$

$$= \int_{E_1} f \, d\mu + \int_{E_2} f \, d\mu + \dots + \int_{E_n} f \, d\mu$$

$$= \sum_{k=1}^{n} \int_{E_k} f \, d\mu.$$

Chapter Seven

Applications of Lebesgue Integration

In this chapter, we introduce some mathematical applications of the Lebesgue integration .

7.1 Convergence of the Lebesgue integral

In this section, we give convergence theorems for Lebesgue integrals . Also,we give some related examples and consequences.

Let (X, F, m) be a measure space and $E \in F$. Let $f : X \to \mathbb{R}$ be a measurable function and Let (f_n) be a sequence of measurable functions defined on E such that

$$\lim_{n \to \infty} f_n(x) = f(x) \quad (x \in E).$$

In general , is not true that

$$\lim_{n \to \infty} \int_E f_n \, dm = \int_E \lim_{n \to \infty} f_n \, dm$$

$$= \int_E f \, dm.$$

For example :

Let $E = [0,1]$ and define the sequence of functions f_n by :

when $0 \le x \le \frac{1}{n}$ the graph of f_n consists of the sides of the triangle with altitude n and base $[0,1]$. when $\frac{1}{n} \le x \le 1$, then $f_n = 0$.

Since $f_n \to 0$ on $[0,1]$, so $\int_0^1 \lim_{n \to \infty} f_n(x)\, dx = 0$.

We have

$$\int_0^1 f_n(x)\, dx = \frac{1}{2}\left(\frac{1}{n}\right)(n)$$

$$= \frac{1}{2}.$$

It follows that $\lim_{n \to \infty} \int_0^1 f_n(x)\,dx = \frac{1}{2}$.

Thus $\int_0^1 \lim_{n \to \infty} f_n(x)\,dx \neq \lim_{n \to \infty} \int_0^1 f_n(x)\,dx$.

Notation

Let X be a non-empty set. Let $f : X \to \mathbb{R}$ and let (f_n) be a sequence of functions defined on X.

The notation $f_n(x) \nearrow f(x)$ $(n \to \infty)$ on X means that

$$f_n(x) \le f_{n+1}(x) \quad \text{for all } n \text{ and } x \in X \quad (\text{Monotonicity}),$$

and

$$f(x) = \lim_{n \to \infty} f_n(x).$$

We have the following properties:

Let $f_n(x) \nearrow f(x)$ and $g_n(x) \nearrow g(x)$ as $n \to \infty$ and for all $x \in X$ and let $h : X \to \mathbb{R}$. Let (a_n) be a sequence of positive real constants and let a be a positive real constant.

Then

(i) $f_n(x) + g_n(x) \nearrow f(x) + g(x)$

(ii) $f_n(x) - h \nearrow f(x) - h$

(iii) If $a_n \nearrow a$, then $a_n f_n(x) \nearrow a f(x)$.

Let X be a non-empty set. Let $g : X \to \mathbb{R}$ and let (g_n) be a sequence of functions defined on X.

The notation $g_n(x) \searrow g(x)$ on X means that

$$g_{n+1}(x) \le g_n(x) \quad \text{for all } n \text{ and } x \in X,$$

and

$$g(x) = \lim_{n \to \infty} g_n(x).$$

We have the following properties :

Let $g_n(x) \downarrow g(x)$. Then

$$\text{(i)} \quad -g_n(x) \uparrow -g(x)$$

$$\text{(ii)} \quad h - g_n(x) \uparrow h - g(x).$$

Theorem 7.1.1 [3]

Let (X, F) be a measurable space and let f be a non-negative bounded measurable function on X. Then there is a sequence of non-negative simple functions (s_n) such that $s_n(x) \uparrow f(x)$ as $n \to \infty$ and for all $x \in X$.

Theorem 7.1.2 (Monotone Convergence Theorem)

Let (X, F, m) be a measure space and $E \in F$. Let (f_n) be a sequence of non-negative measurable functions defined on E such that $f_n(x) \uparrow f(x)$. Then

$$\lim_{n \to \infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof

Since $0 \le f_n(x) \le f(x)$ for all n, so

$$\int_E f_n \, dm \le \int_E f \, dm \quad \text{(Lemma 6.2.2)}.$$

It follows that

$$\lim_{n \to \infty} \int_E f_n \, dm \le \int_E f \, dm \quad \to \quad \text{(i)}$$

Let $0 < a < 1$ and $0 \le h \le f$ be a simple function.

Set

$$E_n = \{ x \in E : f_n(x) \ge a h(x) \}.$$

Then $E_1 \subset E_2 \subset E_3 \subset \ldots$ and E_n are measurable sets (Theorem 5.19 (ii)).

Also, we have $\bigcup_{n=1}^{\infty} E_n = E$.

Therefore

$$\int_{E_n} \alpha h \, d\mu \le \int_{E_n} f_n \, d\mu$$

$$\le \int_{E} f_n \, d\mu \quad \text{(Proposition 6.2 .11)}.$$

So

$$\lim_{n \to \infty} \int_{E_n} \alpha h \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu.$$

It follows from Theorem 6.1.11 that

$$\int_{E} \alpha h \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu,$$

and so

$$\alpha \int_{E} h \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu.$$

Since α is an arbitrary in $(0,1)$, taking

$$\alpha = 1 - \frac{1}{2n}.$$

Therefore

$$\left(1 - \frac{1}{2n}\right) \int_{E} h \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu.$$

Letting $n \to \infty$, so we have

$$\int_{E} h \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu.$$

Taking a supremum over all h such that $0 \le h \le f$, so

$$\sup_{h \le f} \left(\int_{E} h \, d\mu \right) \le \lim_{n \to \infty} \int_{E} f_n \, d\mu,$$

and hence

$$\int_{E} f \, d\mu \le \lim_{n \to \infty} \int_{E} f_n \, d\mu \quad \to \quad (\text{ii})$$

It follows from (i) and (ii) that

$$\lim_{n \to \infty} \int_{E} f_n \, d\mu = \int_{E} f \, d\mu.$$

Remark 7.1.1

The monotonicity condition in the monotone convergence theorem cannot be dropped.

For example :

Let $E = [0,1]$.

Let $\mathcal{F}$ be the $\sigma-$field of all open sets in $[0,1]$.

Let $\mu = m$ (the Lebesgue measure) and define

$$f_n = n\, \chi_{(0,\frac{1}{n})} \quad (n = 1,2,3,\dots).$$

Then (f_n) is a decreasing sequence of non-negative measurable functions.

Clearly $\lim_{n\to\infty} f_n = 0$ and so $\int_E \lim_{n\to\infty} f_n \, d\mu = 0$.

We have

$$\int_E f_n \, d\mu = n\, m\left((0, \tfrac{1}{n}) \right)$$

$$= n \left(\frac{1}{n} \right)$$

$$= 1,$$

and hence

$$\lim_{n\to\infty} \int_X f_n \, d\mu = 1.$$

Thus

$$\int_E \lim_{n\to\infty} f_n \, d\mu < \lim_{n\to\infty} \int_E f_n \, d\mu.$$

The next two corollaries are consequences of Monotone Convergence Theorem.

Corollary 7.1.3

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. let f be a non-negative bounded measurable function on E. Let (f_n) be a sequence of measurable *functions defined on* E *such that* $f_n(x) \uparrow f(x)$. *Let* $f_n \ge h$ *for all* n *and* $\int_E h \, d\mu > -\infty$. *Then*

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof

Let $f_n(x) \uparrow f(x)$. Then

$$f_n(x) - h \uparrow f(x) - h.$$

Since $f_n \ge h$, it follows that $f_n - h \ge 0$.

Since $(f_n - h)$ is a sequence of non-negative measurable functions and $f_n - h \uparrow f - h$, so Monotone Convergence Theorem 7.1.2 gives us

$$\int_E (f - h) \, d\mu = \lim_{n \to \infty} \int_E (f_n - h) \, d\mu.$$

Therefore

$$\int_E f \, d\mu - \int_E h \, d\mu = \lim_{n \to \infty} \left(\int_E f_n \, d\mu - \int_E h \, d\mu \right) \quad (\text{Lemma } 6.2.8)$$

$$= \lim_{n \to \infty} \int_E f_n \, d\mu - \int_E h \, d\mu.$$

Thus

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E f_n \, d\mu.$$

Corollary 7.1.4

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$. *Let* g *be a non-negative bounded measurable function on* E. *Let* (g_n) *be a sequence of measurable functions defined on* E *such that* $g_n(x) \downarrow g(x)$. *Let* $g_n \le h$ *for all* n *and* $\int_E h \, d\mu < \infty$. *Then*

$$\lim_{n \to \infty} \int_E g_n \, d\mu = \int_E g \, d\mu.$$

Proof

Let $g_n(x) \uparrow g(x)$. Then

$$h - g_n(x) \downarrow h - g(x).$$

Since $g_n \le h$, it follows that $h - g_n \ge 0$.

Since $(h - g_n)$ is a sequence of non-negative measurable functions and $h - g_n \downarrow h - g$, so Monotone Convergence Theorem gives us

$$\int_E (h - g)\,d\mu = \lim_{n \to \infty} \int_E (h - g_n)\,d\mu.$$

Therefore

$$\int_E h\,d\mu - \int_E g\,d\mu = \lim_{n \to \infty} \left(\int_E h\,d\mu - \int_E g_n\,d\mu \right).$$

So

$$\int_E h\,d\mu - \int_E g\,d\mu = \int_E h\,d\mu - \lim_{n \to \infty} \int_E g_n\,d\mu,$$

and hence

$$\int_E g\,d\mu = \lim_{n \to \infty} \int_E g_n\,d\mu.$$

Monotone Convergence Theorem allows to prove linearity of the Lebesgue integral for non-negative measurable functions.

Theorem 7.1.5

Let (X, F, μ) be a measure space and $E \in F$. Let f be a non-negative bounded measurable function on E and let a be a positive real constant. Then

$$\int_E af\,d\mu = a \int_E f\,d\mu.$$

Proof

Let f be a non-negative bounded measurable function on E. There exists a sequence of non-negative simple functions (s_n) such that

$$s_n \uparrow f \quad (\text{Theorem 7.1.1}).$$

It follows from Monotone Convergence Theorem that

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E s_n \, d\mu .$$

Choose a positive sequence (a_n) of positive real constants and a is a positive real constant such that $a_n \uparrow a$.

It follows that $a_n s_n \uparrow a f$.

Again, Monotone Convergence Theorem gives us

$$\begin{aligned}
\int_E a f \, d\mu &= \lim_{n \to \infty} \int_E a_n s_n \, d\mu \\
&= \lim_{n \to \infty} \left(a_n \int_E s_n \, d\mu \right) \\
&= \left(\lim_{n \to \infty} a_n \right) \left(\lim_{n \to \infty} \int_E s_n \, d\mu \right) \\
&= a \int_E f \, d\mu .
\end{aligned}$$

Theorem 7.1.6

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let f, g be non-negative measurable functions on E. Then

$$\int_E (f + g) \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu .$$

Proof

Let f, g be non-negative measurable functions on E. There exist two sequences of non-negative simple functions (s_n) and (t_n) such that

$s_n \uparrow f$ and $t_n \uparrow g$ (Theorem 7.1.1).

It follows from Monotone Convergence Theorem that

$$\int_E f \, d\mu = \lim_{n \to \infty} \int_E s_n \, d\mu ,$$

and

$$\int_E g \, d\mu = \lim_{n \to \infty} \int_E t_n \, d\mu .$$

We have $s_n + t_n \uparrow f + g$.

It follows from Monotone Convergence Theorem that

$$\int_E (f+g)\,d\mu = \lim_{n\to\infty} \int_E (s_n + t_n)\,d\mu$$

$$= \lim_{n\to\infty} \left(\int_E s_n\,d\mu + \int_E t_n\,d\mu \right)$$

$$= \lim_{n\to\infty} \int_E s_n\,d\mu + \lim_{n\to\infty} \int_E t_n\,d\mu.$$

Thus

$$\int_E (f+g)\,d\mu = \int_E f\,d\mu + \int_E g\,d\mu.$$

Remark 7.1.2

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E_1, E_2, \ldots \in \mathcal{F}$ with $E_i \cap E_j = \emptyset$ $(i \neq j)$. Let f be a bounded measurable function.

Then

$$f^+ \chi_{\bigcup_{k=1}^{n} E_k} \uparrow f^+ \chi_{\bigcup_{k=1}^{\infty} E_k},$$

and

$$f^- \chi_{\bigcup_{k=1}^{n} E_k} \uparrow f^- \chi_{\bigcup_{k=1}^{\infty} E_k}.$$

Theorem 7.1.7

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E_1, E_2, \ldots \in \mathcal{F}$ with $E_i \cap E_j = \emptyset$ $(i \neq j)$. Let f be a bounded measurable function.

Then

$$\int_{\bigcup_{k=1}^{\infty} E_k} f\,d\mu = \sum_{k=1}^{\infty} \int_{E_k} f\,d\mu.$$

Proof

We have

$$\int_{\bigcup_{k=1}^{\infty} E_k} f \, d\mu = \int_X f \chi_{\bigcup_{k=1}^{\infty} E_k} \, d\mu$$

$$= \int_X \left(f \chi_{\bigcup_{k=1}^{\infty} E_k} \right)^+ d\mu - \int_X \left(f \chi_{\bigcup_{k=1}^{\infty} E_k} \right)^- d\mu.$$

By Remark 7.1.2 , we have

$$f^+ \chi_{\bigcup_{k=1}^{n} E_k} \uparrow f^+ \chi_{\bigcup_{k=1}^{\infty} E_k} \quad \text{and} \quad f^- \chi_{\bigcup_{k=1}^{n} E_k} \uparrow f^- \chi_{\bigcup_{k=1}^{\infty} E_k} .$$

It follows from Monotone Convergence Theorem that

$$\int_{\bigcup_{k=1}^{\infty} E_k} f \, d\mu = \lim_{n \to \infty} \int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^+ d\mu - \lim_{n \to \infty} \int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^- d\mu$$

$$= \lim_{n \to \infty} \left(\int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^+ d\mu - \int_X \left(f \chi_{\bigcup_{k=1}^{n} E_k} \right)^- d\mu \right)$$

$$= \lim_{n \to \infty} \int_X f \chi_{\bigcup_{k=1}^{n} E_k} \, d\mu$$

$$= \lim_{n \to \infty} \int_{\bigcup_{k=1}^{n} E_k} f \, d\mu$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \int_{E_k} f \, d\mu \quad \text{(Theorem 6.3.9)}$$

$$= \sum_{k=1}^{\infty} \int_{E_k} f \, d\mu.$$

Theorem 7.1.8 (Fatous Lemma)

Let $(X, \mathcal{F}, \mu)$ *be a measure space and* $E \in \mathcal{F}$. *Let* (f_n) *be a sequence of non-negative measurable functions defined on* E. *Then*

$$\int_E \underline{\lim}\, f_n \, d\mu \le \underline{\lim} \int_E f_n \, d\mu.$$

Proof

We have

$$\underline{\lim}\, f_n(x) = \lim_{n \to \infty} \left(\inf_{k \ge n} f_k(x) \right) \to (\text{i})$$

Set $g_n(x) = \inf_{k \ge n} f_k(x)$.

Then $g_n(x) \le g_{n+1}(x)$ and so $g_n \uparrow \underline{\lim}\, f_n$.

By Monotone Convergence Theorem , we have

$$\lim_{n \to \infty} \int_E g_n \, d\mu = \int_E \underline{\lim}\, f_n \, d\mu \to (\text{ii})$$

Since $0 \le g_n(x) \le f_k(x)$, it follows that

$$\int_E g_n \, d\mu \le \int_E f_k \, d\mu \quad (\text{Lemma } 6.2.2).$$

Taking an infimum over $k \ge n$, we get

$$\int_E g_n \, d\mu \le \inf_{k \ge n} \int_E f_k \, d\mu \to (\text{iii}).$$

We have

$$\underline{\lim} \int_E f_n \, d\mu = \lim_{n \to \infty} \left(\inf_{k \ge n} \int_E f_k \, d\mu \right) \quad (\text{by (i)})$$

$$\ge \lim_{n \to \infty} \int_E g_n \, d\mu \quad (\text{by (iii)})$$

$$= \int_E \underline{\lim}\, f_n \, d\mu \quad (\text{by (ii)}).$$

Thus

$$\int_E \underline{\lim}\, f_n \, d\mu \le \underline{\lim} \int_E f_n \, d\mu.$$

Example 7.1.1

Let $E = \mathbb{R}$ and define the sequence (f_n) defined on E by

$$f_n(x) = \begin{cases} 1 & x \in [n, n+1] \\ 0 & \text{otherwise.} \end{cases}$$

That is, $f_n = \chi_{[n,n+1]}$.

Then $$\int_{\mathbb{R}} f_n \, dm = \int \chi_{[n,n+1]} \, dm$$

$$= m([n, n+1])$$

$$= 1.$$

Therefore

$$\underline{\lim} \int_{\mathbb{R}} f_n \, dm = 1.$$

We have $\lim_{n \to \infty} f_n(x) = 0$. It follows that

$$\lim_{n \to \infty} f_n(x) = \underline{\lim} \, f_n(x) = 0,$$

and hence $\int_{\mathbb{R}} \underline{\lim} \, f_n \, dm = 0.$

Thus $$\int_{\mathbb{R}} \underline{\lim} \, f_n \, dm \le \underline{\lim} \int f_n \, dm.$$

Corollary 7.1.9

Let (X, F, m) be a measure space and $E \in F$. Let (f_n) be a sequence of non-negative measurable functions defined on E such that $f_n \to f$. If there exist a positive constant M such that $\int_E f_n \, dm \le M$ for all n, then

$$\int_E f \, dm \le M.$$

Proof

We have

$$\int_E \underline{\lim}\, f_n \, d\mu \le \underline{\lim} \int_E f_n \, d\mu \quad (\text{ Fatous Lemma }).$$

Since $\int_E f_n \, d\mu \le M$ for all n, it follows that $\underline{\lim} \int_E f_n \, d\mu \le M$,

and hence $\int_E \underline{\lim}\, f_n \, d\mu \le M$.

Since $f_n \to f$, so we have $\underline{\lim}\, f_n = \lim f_n = f$.

Hence $\int_E f \, d\mu \le M$.

Theorem 7.1.10 (Lebesgue Dominated Convergence Theorem)

Let $(X, \mathcal{F}, \mu)$ be a measure space and $E \in \mathcal{F}$. Let (f_n) be a sequence of measurable functions defined on E such that $f_n \to f$. Let g be a non-negative measurable function such that $|f_n| \le g$ for all n and $\int_E g \, d\mu < \infty$. Then

$$\lim_{n \to \infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Proof

Let $|f_n| \le g$. Then

$$-g \le f_n \le g,$$

and so $g - f_n \ge 0$ and $f_n + g \ge 0$.

Therefore

$$g - f_n \to g - f$$

and

$$f_n + g \to f + g.$$

So

$$\int_E (g - f)\, d\mu = \int_E \lim_{n \to \infty} (g - f_n)\, d\mu$$

$$= \int_E \underline{\lim}\, (g - f_n)\, d\mu$$

Since $g - f_n$ are non-negative measurable functions , so

$$\int_E (g - f)\, d\mu \le \underline{\lim} \int_E (g - f_n)\, d\mu \text{ (Fatous Lemma)}$$

$$= \int_E g\, d\mu - \overline{\lim} \int_E f_n\, d\mu.$$

So we have

$$\int_E g\, d\mu - \int_E f\, d\mu \le \int_E g\, d\mu - \overline{\lim} \int_E f_n\, d\mu,$$

and hence

$$\int_E f\, d\mu \ge \overline{\lim} \int_E f_n\, d\mu \rightarrow \text{ (i)}$$

Similarly, Since $f_n + g$ are non-negative measurable functions, so

$$\int_E (g + f)\, d\mu = \int_E \lim_{n \to \infty} (g + f_n)\, d\mu$$

$$= \int_E \underline{\lim}\, (g + f_n)\, d\mu$$

$$\le \underline{\lim} \int_E (g + f_n)\, d\mu.$$

So

$$\int_E g\, d\mu + \int_E f\, d\mu \le \int_E g\, d\mu + \underline{\lim} \int_E f_n\, d\mu,$$

and hence

$$\int_E f\,dm \le \underline{\lim} \int_E f_n\,dm \qquad (\text{ii})$$

It follows from (i) and (ii) that

$$\int_E f\,dm \le \underline{\lim} \int_E f_n\,dm \le \overline{\lim} \int_E f_n\,dm \le \int_E f\,dm.$$

Therefore

$$\int_E f\,dm = \underline{\lim} \int_E f_n\,dm = \overline{\lim} \int_E f_n\,dm.$$

Hence

$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E f\,dm.$$

Example 7.1.2

Let $E=[0,1]$ and $f_n(x) = n\sqrt{x}\,e^{-n^2x^2}$ $(n \in \mathbb{N},\ x \in E)$.

We will find the limit of the integral

$$\lim_{n\to\infty} \int_0^1 n\sqrt{x}\,e^{-n^2x^2}\,dx,$$

by using the Lebesgue Dominated Convergence Theorem .

We have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n\sqrt{x}\,e^{-n^2x^2}$$

$$= 0.$$

Then

$$n\sqrt{x}\,e^{-n^2x^2} = \frac{1}{\sqrt{x}}\,n\,x\,e^{-n^2x^2}$$

$$\leq \frac{1}{\sqrt{x}}.$$

Thus

$$n\sqrt{x}\, e^{-n^2x^2} \leq \frac{1}{\sqrt{x}} \quad \text{for all } n,$$

where

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx < \infty .$$

The Lebesgue Dominated Convergence Theorem 7.1.10 applies and

$$\lim_{n\to\infty} \int_0^1 n\sqrt{x}\, e^{-n^2x^2}\, dx = \int_0^1 0\, dx$$

$$= 0.$$

7.2 L^p – Spaces

We introduce L^p spaces for every p $(1 \leq p < \infty)$. An important application of Lebesgue intergation is L^p and these spaces play important roles in functional analysis and its applications .

Definition 7.2.1

Let (X, F, m) be a measure space and $E \in F$. Let f be a measurable function on E and $1 \leq p < \infty$. We define $L^p(E, F, \mu)$ by

$$L^p(E, F, \mu) = \{ f : \int_E |f|^p\, d\mu < \infty \}.$$

We shall give some properties of $L^p(E, F, \mu)$ in the next results .

Lemma 7.2.1

Let $f \in L^p(E, F, \mu)$ and let a be a non-zero constant . Then

$$\alpha f \in L^p(E, F, \mu).$$

Proof

Let $f \in L^p(E, F, \mu)$. Then

$$\int_E |f|^p \, d\mu < \infty .$$

We have αf is a measurable function (Theorem 5. 12).

Then

$$\int_E |\alpha f|^p \, d\mu = \int_E |\alpha|^p |f|^p \, d\mu$$

$$= |\alpha|^p \int_E |f|^p \, d\mu$$

$$< \infty .$$

Hence $\alpha f \in L^p(E, F, \mu)$.

Lemma 7.2.2

Let $f, g \in L^p(E, F, \mu)$. *Then* $f + g \in L^p(E, F, \mu)$.

Proof

Let $f, g \in L^p(E, F, \mu)$. Then

$$\int_E |f|^p \, d\mu < \infty \quad \text{and} \quad \int_E |g|^p \, d\mu < \infty .$$

We have $f + g$ is a measurable function (Theorem 5.14).

Then

$$\int_E |f + g|^p \, d\mu \le \int_E \left(|f| + |g|\right)^p d\mu$$

$$\le \int_E 2^p \left(|f|^p + |g|^p\right) d\mu$$

$$= 2^p \left(\int_E |f|^p \, d\mu + \int_E |g|^p \, d\mu\right)$$

$$< \infty .$$

Hence $f + g \in L^p(E, F, \mu)$.

Corollary 7.2.3

Let $f, g \in L^p(E, F, \mu)$ *and let* a, b *be non-zero constants. Then*

$$\alpha f + \beta g \in L^p(E, F, \mu).$$

Proof

The proof follows from Lemma 7.2.1 and Lemma 7.2.2 .

Remark 7.2.1

Let $a = 1$ and $b = -1$ in Corollary 7.2 .3 . Then

$$f - g \in L^p(E, F, \mu).$$

Theorem 7.2.4

Let $f \in L^p(E, F, \mu)$ *and* $g \le f$. *Then* $g \in L^p(E, F, \mu)$.

Proof

Let $f \in L^p(E, F, \mu)$ and $g \le f$. Then

$$\{x : g(x) > c\} = \{x : c < g(x) \le f(x)\}$$

$$= \{x : c < f(x)\} \in F.$$

Thus g is a measurable function .

Since $g \le f$, so $|g|^p \le |f|^p$ for all 1 £ p < ¥ .

Then

$$\int_E |g|^p \, d\mu \le \int_E |f|^p \, d\mu,$$

and so

$$\int_E |g|^p \, d\mu \le \int_E |f|^p \, d\mu < \infty.$$

Hence $\int_E |g|^p \, d\mu < \infty.$

Thus $g \in L^p(E, F, \mu).$

Lemma 7.2.5

Let $f \in L^p(E, F, \mu)$ *. Then* $|f| \in L^p(E, F, \mu)$.

Proof

Since f is a measurable function, so $|f|$ is measurable (Lemma 5.18).
Also, since $|f| \le |f|^p$ (1 £ p < ¥), it follows that

$$\int_E |f| \, d\mu \le \int_E |f|^p \, d\mu < \infty .$$

Hence

$$\int_E |f| \, d\mu < \infty .$$

Thus $|f| \in L^p(E, F, \mu)$.

In next two theorems, we take $E = [0,1]$ and $p = 2$.

Theorem 7.2.6 [4]

Let $f, g \in L^2[0,1]$. *Then*

$$\int_0^1 |fg| \, d\mu \le \left(\int_0^1 |f|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_0^1 |g|^2 \, d\mu \right)^{\frac{1}{2}} .$$

Theorem 7.2.7

Let $f \in L^2[0,1]$. *Then*

$$\left| \int_0^1 f \, d\mu \right| \le \left(\int_0^1 |f|^2 \, d\mu \right)^{\frac{1}{2}} .$$

Proof

Let $f, g \in L^2[0,1]$. Then

$$\int_0^1 |fg| \, d\mu \le \left(\int_0^1 |f|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_0^1 |g|^2 \, d\mu \right)^{\frac{1}{2}}$$

(Theorem 7.2.6) .

Taking $g(x) = 1$ for all x, we get

$$\int_0^1 |f| \, d\mu \leq \left(\int_0^1 |f|^2 \, d\mu \right)^{\frac{1}{2}} .$$

Since $\left| \int_0^1 f \, d\mu \right| \leq \int_0^1 |f| \, d\mu$ (Proposition 6.3.7) , it follows that

$$\left| \int_0^1 f \, d\mu \right| \leq \left(\int_0^1 |f|^2 \, d\mu \right)^{\frac{1}{2}} .$$

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الخلاصة

في هذه الرسالة سوف ندرس ونستعرض المفاهيم الاتية :

مقياس لباق للفئات و مجموعة الفئات المقاسة و مجموعة الفئات μ^* – المقاسة و مجموعة الدوال المقاسة و تكامل لباق .

سوف نقوم بعرض بعضا من خواص المفاهيم السابقة . وايضا سوف نستعرض بعض الحقائق الاساسية والارتباطات المختلفة والامثلة المتعلقة و تطبيقات لتكامل لباق.

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مقدم من

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