Some properties of certain subclasses of p-valent functions defined by a linear derivative operator

Maslina Darus
National University of Malaysia

Available online at http://scik.org
J. Math. Comput. Sci. 1 (2011), No. 1, 66-76
ISSN: 1927-5307

SOME PROPERTIES OF CERTAIN SUBCLASSES OF p-VALENT FUNCTIONS DEFINED BY A LINEAR DERIVATIVE OPERATOR

N. M. MUSTAFA AND M. DARUS*

School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia.

Abstract. In the present paper, we introduce new classes of p-valent functions defined by using a generalized linear derivative operator with negative coefficients in the unit disk. The results presented here include coefficient estimates, extreme points and distortion properties for the aforementioned classes.

Key words. p-valent functions, starlike, convex, distortion theorems, linear derivative operator.

AMS Mathematics Subject Classification (2000): 30C45.

1. Definition and Preliminaries

Let A_p denote the class of functions of the form :

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathbb{N}). \tag{1.1}$$

*Corresponding author

E-mail addresses: nmma1975@yahoo.com (N. M. Mustafa), maslina@ukm.my (M. Darus)

Received November 27, 2011

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in A_p$ is called p-valent starlike of order β and type γ, if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\gamma} \right| < \beta, \tag{1.2}$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in N$. We denote by $S^*(p,\gamma,\beta)$ the class of p-valent starlike functions of order γ and type β. A function $f \in A_p$ is called p-valent convex functions of order β and type γ, if it satisfies

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\gamma} \right| < \beta, \tag{1.3}$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in N$. We denote by $K(p,\gamma,\beta)$ the class of p-valent convex functions of order γ and type β.

From (1.2) and (1.3), we note that: $f(z) \in K(p,\gamma,\beta)$ if, and only if,

$$\frac{zf'}{p} \in S^*(p,\gamma,\beta).$$

The classes $S^*(p,\gamma,\beta)$ and $K(p,\gamma,\beta)$ were considered by Aouf [2] and Hossen [3]. For $\beta = 1$, reduced to the class $S^*(p,\gamma,1) = S^*(p,\gamma)$ which was studied by Patil and Thakare [4], and the class $K(p,\gamma,1) = K(p,\gamma)$ given by Owa [5].

Let T_p denote the subclass of A_p consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathbb{N}). \tag{1.4}$$

We denote by $T^*(p,\gamma,\beta)$ and $C(p,\gamma,\beta)$, the classes obtained by taking intersections, respectively, of the classes $S^*(p,\gamma,\beta)$ and $K(p,\gamma,\beta)$ with the class T_p. Thus we have

$$T^*(p,\gamma,\beta) = S^*(p,\gamma,\beta) \cap T_p,$$

and

$$C(p,\gamma,\beta) = K(p,\gamma,\beta) \cap T_p.$$

The classes $T^*(p,\gamma,\beta)$ and $C(p,\gamma,\beta)$ were studied by Aouf [2] and Hossen [3]. In particular, the classes $T^*(p,\gamma,1)=T^*(p,\gamma)$ and $C(p,\gamma,1)=C(p,\gamma)$ were introduced by Owa [5]. Also the classes $T^*(1,\gamma,1)=T^*(\gamma)$ and $C(1,\gamma,1)=C(\gamma)$ were studied by Silverman [6].

For functions $f\in A_p$, given by (1.1), and g given by

$$g(z)=z^p+\sum_{k=p+1}^{\infty}b_kz^k, \qquad (p\in\mathbb{N}),$$

the Hadamard product(or convolution)of functions f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kb_kz^k=(g*f)(z), \qquad (p\in\mathbb{N}).$$

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(x)_k=$

$$\begin{cases}1 & for \quad k=0,\\ x(x+1)(x+2)...(x+k-1) & for \quad k\in\mathbb{N}=\{1,2,3,...\}.\end{cases}$$

The authors in [1] have recently introduced a new generalized linear derivative operator $D_p^{\alpha,\delta}(\mu,q,\gamma)$, as the following:

Definition 1.1. *For $f\in A_p$, the linear operator $D_p^{\alpha,\delta}(\mu,q,\gamma)$ is defined by $D_p^{\alpha,\delta}(\mu,q,\gamma):A_p\to A_p$ as:*

$$D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)=z^p+\sum_{k=p+1}^{\infty}(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}c(\delta,k)a_kz^k, \qquad (1.5)$$

where $\lambda,\mu,q\geq 0,\ k,\delta,\alpha\in\mathbb{N}_0=\{0,1,2...\}$,

and $c(\delta,k)=z^p+\sum_{k=1}^{\infty}\frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)}z^k$.

Next we define the following new subclasses of p-valent functions as follows:

Definition 1.2. *Let $f \in T_p$ be given by (1.4). Then f is said to be in the class $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$ if, and only if,*

$$\left| \frac{\frac{z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)} - p}{\frac{z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)} + p - 2\gamma} \right| < \beta,$$

where $D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)$ is given by (1.5) and $\lambda,\mu,q \geq 0$, $k,\delta,\alpha \in \mathbb{N}_0 = \{0,1,2...\}$, $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in N$.

Further, a function $f \in T_p$ is said to be in the class $C_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$ if, and only if,

$$\frac{zf'}{p} \in T_p^{\alpha,\delta}(\mu,q,\gamma,\beta).$$

We note that, by specializing the parameters $\alpha,\delta,\mu,\lambda,\beta$ and p, we shall obtain the following subclasses which were studied by various authors:

1. For $\alpha = \delta = \mu = 0$ we get $T_p^{0,0}(0,q,\gamma,\beta) = T^*(p,\gamma,\beta)$, is the class of p-valent starlike function of order γ and type β which was studied by Aouf [2] and Hossen [3].

2. For $\alpha = \delta = \mu = 0$ and $p = 1$, we have $T_1^{0,0}(0,q,\gamma,\beta) = S^*(\gamma,\beta)$, is the class of starlike function of order γ and type β which was studied by Gupta and Jain [7].

3. For $\alpha = \delta = \mu = 0$ and $\beta = 1$, we obtain the class $T_p^{0,0}(0,q,\gamma,1) = T^*(p,\gamma)$, which was introduced by Owa [5].

4. For $\alpha = \delta = \mu = 0$, $p = 1$ and $\beta = 1$ we obtain the class $T_1^{0,0}(0,q,\gamma,1) = T^*(\gamma)$, which was studied by Silverman [6].

5. For $\alpha = \delta = q = 0$,$\mu = 1$ and $p = 1$, we have the class $C_1^{0,0}(1,0,\gamma,\beta) = C^*(\gamma,\beta)$, which was studied by Gupta and Jain [7].

6. For $\alpha = \delta = q = 0$,$\mu = 1$, we have the class $C_p^{0,0}(1,0,\gamma,\beta) = C(p,\gamma,\beta)$, is the class of p-valent convex function of order γ and type β, studied by Aouf [2] and Hossen [3].

7. For $\alpha = \delta = q = 0$,$\mu = 1$,and $\beta = 1$, we have the class $C_p^{0,0}(1,0,\gamma,1) = C(p,\gamma)$, studied by Owa [5].

8. For $\alpha=\delta=q=0$,$\mu=1$,$\beta=1$, and $p=1$, we obtain the class $C_1^{0,0}(1,0,\gamma,1)=C(\gamma)$, studied by Silverman [6].

2. Coefficient Estimates

Theorem 2.1. *A function f belongs to the class $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$ if, and only if,*

$$\sum_{k=p+1}^{\infty}\left(((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k\right)\leq 2\beta(p-\gamma). \quad (2.1)$$

Proof: Let the function f be in the class $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$. Then we have

$$\left|\frac{\frac{z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)}-p}{\frac{z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)}+p-2\gamma}\right|=\left|\frac{\frac{pz^p-\sum_{k=p+1}^{\infty}(k)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k}{z^p-\sum_{k=p+1}^{\infty}(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k}-p}{\frac{pz^p-\sum_{k=P+1}^{\infty}(k)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k}{z^p-\sum_{k=p+1}^{\infty}(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k}+p-2\gamma}\right|\leq\beta.$$

Since $|Re(z)|\leq|z|$ for all z, we have

$$\Re\left\{\frac{\sum_{k=p+1}^{\infty}(k-p)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k}{-\sum_{k=p+1}^{\infty}(k+p-2\gamma)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k+(2p-2\gamma)}\right\}\leq\beta.$$

Choosing values of z on the real axis, so that $\frac{z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)}$ is real, and letting $z\to 1^-$, through real axis, we get

$$\sum_{k=p+1}^{\infty}(k-p)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)}a_kz^k\leq$$

$$-\beta(\sum_{k=p+1}^{\infty}(k+p-2\gamma)(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_kz^k+\beta(2p-2\gamma)),$$

which implies the assertion (2.1). Conversely, let the inequality (2.1) holds true, then

$$\left|z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)-p(D_p^{\alpha,\delta}(\mu,q,\gamma)f(z))\right|-\beta$$

$$\left|z(D_p^{\alpha,\delta}(\mu,q,\gamma)f)'(z)+(p-2\gamma)D_p^{\alpha,\delta}(\mu,q,\gamma)f(z)\right|,$$

$$\sum_{k=p+1}^{\infty}\left(((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}\right)-\beta(2p-2\gamma)\leq 0,$$

by the assumption. This implies that $f\in T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$.

Corollary 2.1. *Let the function f be in the class $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, then*

$$a_k \leq \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}. \tag{2.2}$$

The result (2.2) is sharp for the function f of the form

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}z^k. \tag{2.3}$$

By using the same arguments as in the proof of Theorem 2.1, we can establish the next theorem.

Theorem 2.2. *A function f belongs to the subclass $C_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, if, and only if,*

$$\sum_{k=p+1}^{\infty}\left(k[(k-p)+\beta(k+p-2\gamma)](\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_k z^k\right) \leq 2\beta p(p-\gamma),$$

Corollary 2.2. *Let the function f be in the class $C_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$. Then*

$$a_k \leq \frac{2\beta p(p-\gamma)}{k[(k-p)+\beta(k+p-2\gamma)](\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}},$$

with equality only for functions of the form

$$f(z) = z^p - \frac{2\beta p(p-\gamma)}{k[(k-p)+\beta(k+p-2\gamma)](\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}z^k.$$

3. Distortion Properties

In this section, we obtain distortion bounds for the classes $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$ and $C_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$.

Theorem 3.1. *If $f \in T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, then*

$$|f(z)| \geq r^p - \frac{2\beta(p-\gamma)}{(1+\beta(1+2p-2\gamma))(\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1} \tag{3.1}$$

$$\leq r^p + \frac{2\beta(p-\gamma)}{(1+\beta(1+2p-2\gamma))(\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1}, \tag{3.2}$$

and

$$|f'(z)| \geq pr^{p-1} - \frac{2\beta(p-\gamma)(p+1)}{(1+\beta(1+2p-2\gamma))(\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p} \tag{3.3}$$

$$\leq p|z|^{p-1} + \frac{2\beta(p-\gamma)(p+1)}{(1+\beta(1+2p-2\gamma))(\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p}, \tag{3.4}$$

for $z \in \mathbb{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof: Since $f \in T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, and in view of inequality (2.1) of Theorem 2.1, we have

$$(1+\beta(1+2p-2\gamma))(\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}\sum_{k=p+1}^{\infty} a_k \leq$$

$$\sum_{k=p+1}^{\infty}\left(((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{k-p}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}a_k z^k\right) \leq 2\beta(p-\gamma),$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{2\beta(p-\gamma)}{(1+\beta(1+2p-2\gamma))(\frac{p+11}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}. \tag{3.5}$$

Since

$$r^p - r^{p+1}\sum_{k=p+1}^{\infty} a_k \leq |f(z)| \leq r^p + r^{p+1}\sum_{k=p+1}^{\infty} a_k, \tag{3.6}$$

on using (3.5) and (3.6), we easily arrive at the desired results of (3.2) and (3.1). Furthermore, we observe that

$$pr^{p-1}-(p+1)r^p\sum_{k=p+1}^{\infty}a_k\le|f'(z)|\le pr^{p-1}+(p+1)r^p\sum_{k=p+1}^{\infty}a_k, \tag{3.7}$$

On using (3.5) and (3.7), we easily arrive at the desired results of (3.3) and (3.4). Finally, we can see that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp for the function,

$$f(z)=z^p-\frac{2\beta(p-\gamma)}{(1+(1+2p-2\gamma))(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}.$$

Similarly, we can prove the following theorem.

Theorem 3.2. *If $f\in C_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, then*

$$|f(z)|\ge r^p-\frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}r^{p+1}$$

$$\le r^p+\frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}r^{p+1},$$

and

$$|f'(z)|\ge pr^{p-1}-\frac{2\beta p(p-\gamma)(p+1)}{[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}r^{p}$$

$$\le pr^{p-1}+\frac{2\beta p(p-\gamma)(p+1)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}r^{p},$$

for $z\in\mathbb{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

4. Extreme Points

Theorem 4.1. *Let $f_p(z)=z^p$ and,*

$$f_k(z)=z^p-\frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^{\alpha}(1+\frac{(k-p)}{p+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}z^k.$$

Then f is in the class $T_p^{\alpha,\delta}(\mu,q,\gamma,\beta)$, if ,and only if, it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1. \tag{4.1}$$

Proof: Let $f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z)$

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k z^k.$$

Then, in view of (4.1), it follows that

$$\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \times$$

$$\left\{ \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k \right\} = \sum_{k=1}^{\infty} \omega_k = 1 - \omega_1 \leq 1.$$

Thus $f \in T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$.

Conversely, assume that a function f defined by (1.4) belongs to class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$. Then

$$a_k \leq \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}.$$

We set

$$\omega_k = \frac{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}{2\beta(p-\gamma)},$$

and $\omega_k = 1 - \sum_{k=1}^{\infty} \omega_k$. Then we have $f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z)$, and hence completes the proof.

Similarly, we can prove the following result:

Theorem 4.2. *Let $f_p(z) = z^p$ and,*

$$f_k(z) = z^p - \frac{2\beta p(p-\gamma)}{k[(k-p)+\beta(k+p-2\gamma)](\frac{k}{p})^\alpha(1+\frac{(k-p)}{p+q}\lambda)^\mu \frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)}} z^k.$$

Then f is in the class $C_p^{\alpha,\delta}(\mu, q, \gamma, \beta.$, if, and only if, it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1.$$

Many other work on p-valent functions related to derivative operator and integral operator can be read in [8]-[10] and [11], respectively.

Acknowledgement The work here is supported by MOHE: UKM-ST-06-FRGS0244-2010.

References

[1] N.Mustafa and M. Darus, Differential subordination and superordination for new linear derivative operator. Int. J. Pure and Appl. Math.**70(6)** (2011), 825-835.

[2] M. K. Aouf, A generalization of multivalent functions with negative coefficients, J. Korean Math. Soc. **25(1)**(1988), 53-66.

[3] H. M. Hossen, Quasi-Hadamard product of certain p-valent functions, Demonstration Math. **33(2)**(2000), 277-281.

[4] D. A. Partil and N. K. Thakare, On convex hulls and extreme points of p-valent starlike and convex classes with applications, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.), **27(75)**(1983), 145-160.

[5] S. Owa, On certain classes of p-valent functions with negative coefficients, Bull. Belg. Math. Soc. Simon Stevin, **59**(1985), 385-402.

[6] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc. **51**(1975), 109-116.

[7] V. P. Gupta and P. K. Jain, Certain classes of univalent functions with negative coefficients, Bull. Austral. Math. Soc.**14**(1976), 409-416.

[8] F. Ghanim and M. Darus, On certain subclasses of multivalent starlike functions. Int. J. Pure Appl. Math. **60(3)** (2010), 299-310.

[9] F. Ghanim and M. Darus, New subclass of multivalent hypergeometric meromorphic functions. Int. J. Pure Appl. Math. **61(3)**(2010), 269-280.

[10] F. Ghanim and M. Darus, Some results of p-valent meromorphic functions defined by a linear operator, Far East J. Math. Sci. (FJMS), **44(2)** (2010), 155-165.

[11] A. Mohammed, M. Darus, D. Breaz, B.A. Frasin, A note on integral operators of p-valent functions, Proc. Pakistan Academy of Sciences, **47(4)** (2010), 227-235.