Far East Journal of Mathematical Sciences (FJMS)
Volume …, Number …, 2012, Pages …
Available online at http://pphmj.com/journals/fjms.htm
Published by Pushpa Publishing House, Allahabad, INDIA

ON SUBCLASSES OF *p*-VALENT FUNCTIONS DEFINED BY A DERIVATIVE OPERATOR

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Abstract

In the present paper, we study certain subclasses $T_p^{\alpha, \delta}(\mu, q, \lambda, \gamma, \beta)$ and $C_p^{\alpha, \delta}(\mu, q, \lambda, \gamma, \beta)$ of analytic *p*-valent functions with negative coefficients in the unit disc. The results presented here include the modified Hadamard product, the radii of close-to-convexity, starlikeness and convexity for functions belonging to the above mentioned subclasses.

1. Definition and Preliminaries

Let A_p denote the class of functions of the form

2010 Mathematics Subject Classification: 30C45.

Keywords and phrases: *p*-valent functions, coefficient, Hadamard product, close-to-convexity, starlikeness, convexity.

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Received July 21, 2012

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}),$$

which is analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in A_p$ is called *p-valent starlike* of order β and type γ if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\gamma} \right| < \beta, \tag{1.1}$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $S^*(p, \gamma, \beta)$ the class of *p*-valent starlike functions of order γ and type β.

A function $f \in A_p$ is called *p-valent convex* of order γ and type β if it satisfies

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\gamma} \right| < \beta, \tag{1.2}$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $K(p, \gamma, \beta)$ the class of *p*-valent convex functions of order γ and type β.

From (1.1) and (1.2), we note that

$$f(z) \in K(p, \gamma, \beta) \text{ if and only if } \frac{zf'(z)}{p} \in S^*(p, \gamma, \beta).$$

The classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ were considered by Aouf [3] and Hossen [4]. For $\beta = 1$, the class $S^*(p, \gamma, 1) = S^*(p, \gamma)$ was studied by Patil and Thakare [5], and the class $K(p, \gamma, 1) = K(p, \gamma)$ was introduced by Owa [6].

Let T_p denote the subclass of A_p consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0,\ p \in \mathbb{N}). \tag{1.3}$$

We denote by $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ with the class T_p. Thus, we have

$$T^*(p, \gamma, \beta) = S^*(p, \gamma, \beta) \cap T_p$$

and

$$C(p, \gamma, \beta) = K(p, \gamma, \beta) \cap T_p.$$

The classes $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$ were studied by Aouf [3] and Hossen [4]. In particular, the classes $T^*(p, \gamma, 1) = T^*(p, \gamma)$ and $C(p, \gamma, 1) = C(p, \gamma)$ were introduced by Owa [6]. Also, the classes $T^*(1, \gamma, 1) = T^*(\gamma)$ and $C(1, \gamma, 1) = C(\gamma)$ were studied by Silverman [7].

The authors in [1] have, recently, introduced a new generalized derivative operator $D_p^{\alpha, \delta}(\mu, q, \lambda)$ as follows:

Definition 1.1. For $f \in A_p$, the operator $D_p^{\alpha, \delta}(\mu, q, \lambda)$ is defined by $D_p^{\alpha, \delta}(\mu, q, \lambda) : A_p \to A_p$ as the following:

$$D_p^{\alpha, \delta}(\mu, q, \lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^{\alpha} \left(1 + \frac{k-p}{p+q}\lambda\right)^{\mu} c(\delta, k) a_k z^k, \tag{1.4}$$

where $\lambda, \mu, q \geq 0,\ k, \delta, \alpha \in \mathbb{N}_0$ and

$$c(\delta, k) = z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(k+\delta)}{(k-p)!\,\Gamma(p+\delta)} z^k.$$

Also, the authors in [2] have, recently, stated new subclasses of analytic functions with negative coefficients given as follows:

Definition 1.2. For $f \in T_p$ is said to be in the class $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ if and only if

$$\left| \frac{\dfrac{z(D_p^{\alpha,\delta}(\mu, q, \lambda) f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)} - p}{\dfrac{z(D_p^{\alpha,\delta}(\mu, q, \lambda) f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)} + p - 2\gamma} \right| < \beta,$$

where $D_p^{\alpha,\delta}(\mu, q, \lambda) f(z)$ is given by (1.4) and $\lambda, \mu, q \geq 0,\ k, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, ...\},\ 0 \leq \gamma < p,\ 0 < \beta \leq 1$ and $p \in \mathbb{N}$.

Further,

$$f \in C_p^{\alpha,n}(\mu, q, \lambda, \gamma, \beta) \text{ if and only if } \frac{zf'}{p} \in T_p^{\alpha,n}(\mu, q, \lambda, \beta).$$

We note that, by specializing the parameters α, δ, μ, λ, β and p, we obtain the following subclasses which were studied by various authors:

1. For $\alpha = \delta = \mu = 0$, we obtain $T_p^{0,0}(0, q, \lambda, \gamma, \beta) = T^*(p, \gamma, \beta)$, is the class of p-valent starlike functions of order γ and type β which was studied by Aouf [3] and Hossen [4].

2. For $\alpha = \delta = \mu = 0$ and $p = 1$, we obtain $T_1^{0,0}(0, q, \lambda, \gamma, \beta) = S^*(\gamma, \beta)$, is the class of starlike functions of order γ and type β which was studied by Gupta and Jain [8].

3. For $\alpha = \delta = \mu = 0$ and $\beta = 1$, we obtain the class $T_p^{0,0}(0, q, \lambda, \gamma, 1) = T^*(p, \gamma)$, which was introduced by Owa [6].

4. For $\alpha = \delta = \mu = 0$, $p = 1$ and $\beta = 1$, we obtain the class $T_1^{0,0}(0, q, \lambda, \gamma, 1) = T^*(\gamma)$, which was studied by Silverman [7].

5. For $\alpha = \delta = q = 0,$ $\mu = 1$ and $p = 1,$ we obtain the class $C_1^{0,0}(1, 0, \lambda, \gamma, \beta) = C^*(\gamma, \beta),$ which was studied by Gupta and Jain [8].

6. For $\alpha = \delta = q = 0,$ $\mu = 1,$ we obtain the class $C_p^{0,0}(1, 0, \lambda, \gamma, \beta) = C(p, \gamma, \beta),$ is the class of *p*-valent convex functions of order γ and type β which was studied by Aouf [3] and Hossen [4].

7. For $\alpha = \delta = q = 0,$ $\mu = 1$ and $\beta = 1,$ we obtain the class $C_p^{0,0}(1, 0, \lambda, \gamma, 1) = C(p, \gamma),$ which was studied by Owa [6].

8. For $\alpha = \delta = q = 0,$ $\mu = 1,$ $\beta = 1$ and $p = 1,$ we obtain the class $C_1^{0,0}(1, 0, \lambda, \gamma, 1) = C(\gamma),$ which was studied by Silverman [7].

In [2], sufficient conditions for a function $f(z) \in T_P$ to be in the subclasses $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ and $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta),$ as stated in the following theorems are provided.

Theorem 1.3. *Let the function f belong to the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}a_k\right)$$

$$\leq 2\beta(p-\gamma),$$

the result is sharp for the function f of the form

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!(p+\delta)}}z^k. \tag{1.5}$$

Theorem 1.4. *Let f belong to the subclass* $C_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ *if and only if*

$$\sum_{k=p+1}^{\infty}\left(k[(k-p)+\beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}a_k\right)$$

$$\leq 2\beta p(p-\gamma)$$

with equality only for functions of the form

$$f(z)=z^p-\frac{2\beta p(p-\gamma)}{k[(k-p)+\beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}z^k.$$

2. Modified Hadamard Products

Let the functions $f_j(z)\ (j=1,2)$ be defined by

$$f_j(z)=z^p-\sum_{k=p+1}^{\infty}a_{k,i}z^k \quad (p\in\mathbb{N}). \tag{2.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z)=z^p-\sum_{k=p+1}^{\infty}a_{k,1}a_{k,2}z^k \quad (p\in\mathbb{N}).$$

Theorem 2.1. *Let the functions* $f_j(z)\ (j=1,2)$ *defined by* (2.1) *be in the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then* $(f_1 * f_2)(z)\in T_p^{\alpha,\delta}(\mu, q, \lambda, \omega, \beta)$, *where*

$$\omega \leq p-\frac{2\beta^2(p-\gamma)^2+2\beta(p-\gamma)^2}{(1+\beta(2p+1-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)}-4\beta^2(p-\gamma)^2}.$$

The result is sharp

$$f(z)=z^p-\frac{2\beta(p-\gamma)}{(1+\beta(2p+1-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)}}z^{p+1}.$$

Proof. To prove the theorem, we need to find the largest ω such that

$$\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\omega))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\omega)}$$

$$\cdot\, a_{k,1}a_{k,2} \leq 1,$$

since

$$\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)}$$

$$\cdot\, a_{k,1} \leq 1$$

and

$$\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)}$$

$$\cdot\, a_{k,2} \leq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)}$$

$$\cdot\, \sqrt{a_{k,1}a_{k,2}} \leq 1.$$

Thus, it suffices to show that

$$\frac{((k-p)+\beta(k+p-2\omega))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\omega)}\, a_{k,1}a_{k,2}$$

$$\leq \frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)}\sqrt{a_{k,1}a_{k,2}}.$$

That is,

$$\sqrt{a_{k+p,1}a_{k+p,2}} \leq \frac{(p-\omega)((k-p)+\beta(k+p-2\gamma))}{(p-\gamma)((k-p)+\beta(k+p-2\omega))}.$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}.$$

Consequently, we need only to prove that

$$\frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}$$

$$\leq \frac{(p-\omega)((k-p)+\beta(k+p-2\gamma))}{(p-\gamma)((k-p)+\beta(k+p-2\omega))}$$

or, equivalently, that

$$\omega \leq p - \frac{2\beta(p-\gamma)^2(k-p)(\beta+1)}{((k-p)+\beta(k+p-2\gamma))^2\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \cdot \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)} - 4\beta^2(p-\gamma)^2} \tag{2.2}$$

is an increasing function of k, $k \geq p+1$, letting $k = p+1$ in (2.2), we obtain

$$\omega \leq \phi(p+1)$$

$$\leq p - \frac{2\beta^2(p-\gamma)^2 + 2\beta(p-\gamma)^2}{(1+\beta(2p+1-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)} - 4\beta^2(p-\gamma)^2}$$

which completes the assertion of theorem.

Similarly, we can prove the following results.

Theorem 2.2. *Let the functions* $f_j(z)\,(j = 1, 2)$ *defined by* (2.1) *be in the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then* $(f_1 * f_2)(z) \in C_p^{\alpha,\delta}(\mu, q, \lambda, \chi, \beta)$, *where*

$$\chi \le p - \frac{2p\beta(p-\gamma)^2(\beta+1)}{((p+1)+\beta(1+2p-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{\lambda}{p+q}\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)} - 4\beta^2 p(p-\gamma)^2}.$$

Finally, by taking the function

$$f(z) = z^p - \frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(2p+1-2\gamma)]\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)}} z^{p+1},$$

we can see that the result is sharp.

Theorem 2.3. *Let the functions* $f_j(z)\,(j = 1, 2)$ *defined by* (2.1) *be in the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then the function*

$$h(z) = z^p - \sum_{k=p+1}^{\infty}(a_{k,1}^2 + a_{k,2}^2)z^k \tag{2.3}$$

belongs to the class $T_p^{\alpha,\delta}(\mu, q, \lambda, \xi, \beta)$, *where*

$$\omega \le p - \frac{4\beta^2(p-\gamma)^2(\beta+1)}{\beta(1+\beta(2p+1-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)} - 8\beta^3(p-\gamma)^2}.$$

The result is sharp for the functions.

Proof. By virtue of Theorem 1.3, we obtain

$$\sum_{k=p+1}^{\infty}\left[\frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\,\Gamma(p+\delta)}}{2\beta(p-\gamma)}\right]^2 a_{k,1}^2$$

$$\leq \left[\sum_{k=p+1}^{\infty} \frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \right]^2 \leq 1$$

(2.4)

and

$$\sum_{k=p+1}^{\infty} \left[\frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \right]^2 a_{k,2}^2$$

$$\leq \left[\frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \right]^2 \leq 1. \quad (2.5)$$

It follows from (2.4) and (2.5) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \right]^2$$

$$\cdot (a_{k,1}^2 + a_{k,2}^2) \leq 1.$$

Therefore, we need to find the largest ξ such that

$$\frac{((k-p)+\beta(k+p-2\xi))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\xi)}$$

$$\leq \frac{1}{2} \left[\frac{((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \right]^2,$$

that is,

$$\xi \leq p - \frac{4\beta^2(p-\gamma)^2(k-p)(\beta+1)}{\beta((k-p)+\beta(k+p-2\gamma))^2\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)} - 8\beta^3(p-\gamma)^2}.$$

Since

$$\omega = p - \frac{4\beta^2(p-\gamma)^2(k-p)(\beta+1)}{\beta((k-p)+\beta(k+p-2\gamma))^2\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)} - 8\beta^3(p-\gamma)^2}$$

is an increasing function of *k*, letting $k = p+1,$ we obtain

$$\phi(p+1) = p - \frac{4\beta^2(p-\gamma)^2(\beta+1)}{\beta(1+\beta(2p+1-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu} \cdot frac\Gamma(p+1+\delta)\Gamma(p+\delta) - 8\beta^3(p-\gamma)^2}$$

which completes the proof.

Theorem 2.4. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (2.1) *be in the class* $C_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta).$ *Then the function* $h(z)$ *defined by* (2.3) *belongs to the class* $C_p^{\alpha,\delta}(\mu, q, \lambda, \chi, \beta),$ *where*

$$\chi \leq p - \frac{4p\beta^2(p-\gamma)^2(\beta+1)}{\beta(p+1)(1+\beta(1+2p-2\gamma))^2\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{\lambda}{p+q}\right)^{\mu} \cdot \frac{\Gamma(p+1+\delta)}{p\Gamma(p+\delta)} - 8\beta^3(p-\gamma)^2}.$$

The result is sharp for the functions

$$f(z) = z^p - \frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(2p+1-2\gamma)]\left(\frac{p+1}{p}\right)^{\alpha}\left(1+\frac{1}{p+q}\lambda\right)^{\mu}\frac{\Gamma(p+1+\delta)}{\Gamma(p+\delta)}} z^{p+1}.$$

3. Radii of Starlikeness, and Convexity

In the next theorems, we will find the radius of starlikeness, convexity and close-to-convexity for the class $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$.

Theorem 3.1. *Let the function f be defined by* (1.3) *belonging to the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then f is p-valently close-to-convex of order* ρ $(0 \le \rho < p)$ *in the disk* $|z| < r$, *where*

$$r := \inf_{k \ge p+1} \left(\frac{(p-\rho)\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}\right)}{2k\beta(p-\gamma)} \right)^{\frac{1}{k-1}}. \tag{3.1}$$

The result is sharp with extremal function f given by (1.5).

Proof. Given $f \in T$ and f is p-valently close-to-convex of order ρ in the disc $|z| < r$ if and only if we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \rho, \text{ whenever } |z| < 1. \tag{3.2}$$

For the left hand side of (3.2), we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \le \sum_{k=p+1}^{\infty} k a_k |z|^{k-1}.$$

Then (3.2) is implied by

$$\sum_{k=p+1}^{\infty} \frac{k}{p-\rho} a_k |z|^{k-1} < 1.$$

Using the fact that $f(z) \in T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}\right)}{2\beta(p-\gamma)} \leq 1,$$

it follows that (3.2) is true if

$$\frac{k}{p-\rho}|z|^{k-1} \leq \frac{\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}\right)}{2\beta(p-\gamma)}$$

whenever $|z| < r.$ We obtain

$$r := \inf_{k \geq p+1} \left(\frac{(p-\rho)\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)} a_k\right)}{2k\beta(p-\gamma)} \right)^{\frac{1}{k-1}}.$$

This completes the proof.

Theorem 3.2. *Let the function f be defined by* (1.3) *belonging to the class* $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then*:

(I) *f is p-valently starlike of order* ρ $(0 \leq \rho < 1)$ *in the disc* $|z| < r,$ *that is,*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (|z| < r,\ 0 \leq \rho < p),$$

where

$$r := \inf_{k \geq p+1} \left(\frac{(p-\rho)\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu} \frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)} a_k\right)}{2(k-\rho)\beta(p-\gamma)} \right)^{\frac{1}{k-1}}.$$

(II) *f is p-valently convex of order* ρ $(0 \leq \rho < 1)$ *in the disc* $|z| < r,$ *that is,*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho \quad (|z| < r,\ 0 \le \rho < p),$$

where

$$r := \inf_{k \ge p+1}\left(\frac{\left(p(p-\rho)((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}a_k\right)}{2k(k-p)\beta(p-\gamma)}\right)^{\frac{1}{k-1}}.$$

Each of these results is sharp for the extremal function given by (1.5).

Proof. (I) Given $f \in T$ and f is p-valently starlike of order ρ in the disc $|z| < r$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p - \rho \text{ whenever } |z| < r. \tag{3.3}$$

For the left hand side of (3.3), we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{k=p+1}^{\infty}(k-p)a_k|z|^{k-1}}{1-\sum_{k=p+1}^{\infty}a_n|z|^{k-1}}.$$

Then (3.3) is implied by

$$\sum_{n=2}^{\infty}\frac{k-\rho}{p-\rho}a_k|z|^{k-1} < 1.$$

Using the fact that $T_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$ if and only if

$$\sum_{k=p+1}^{\infty}\frac{\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}\right)}{2\beta(p-\gamma)} \le 1.$$

(3.3) is true for every z in the disc $|z| < r$ if

$$\frac{k-\rho}{p-\rho}|z|^{k-1} \le \frac{\left(((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}\right)}{2\beta(p-\gamma)}.$$

Thus,

$$r := \inf_{k\ge p+1}\left(\frac{\left((p-\rho)((k-p)+\beta(k+p-2\gamma))\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}a_k\right)}{2(k-p)\beta(p-\gamma)}\right)^{\frac{1}{k-1}}.$$

This completes the proof.

(II) Using the fact that f is convex of order ρ if and only if $zf'(z)$ is starlike of order ρ, we can prove (II) using similar methods to the proof of (I).

Similarly, we can prove the following results.

Theorem 3.3. *Let the function* f *be defined by* (1.3) *belonging to the class* $C_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then* f *is* p*-valently close-to-convex of order* ρ $(0 \le \rho < p)$ *in the disc* $|z| < r$, *where*

$$r := \inf_{k\ge p+1}\left(\frac{(p-\rho)[(k-p)+\beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2\beta p(p-\gamma)}\right)^{\frac{1}{k-1}}. \tag{3.4}$$

The result is sharp with extremal function f given by (1.5).

Theorem 3.4. *Let the function* f *be defined by* (1.3) *belonging to the class* $C_p^{\alpha,\delta}(\mu, q, \lambda, \gamma, \beta)$. *Then*:

(I) f *is* p*-valently starlike of order* ρ $(0 \le \rho < 1)$ *in the disc* $|z| < r$, *that is*,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (|z| < r,\ 0 \le \rho < p),$$

where

$$r := \inf_{k \ge p+1} \left(\frac{k(p-\rho)[(k-p)+\beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{(k-p)2\beta p(p-\gamma)} \right)^{\frac{1}{k-1}}.$$

(II) *f is p-valently convex of order* ρ $(0 \le \rho < 1)$ *in the disc* $|z| < r,$ *that is*,

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho \quad (|z| < r,\ 0 \le \rho < p),$$

where

$$r := \inf_{k \ge p+1} \left(\frac{p(p-\rho)[(k-p)+\beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^{\alpha}\left(1+\frac{k-p}{p+q}\lambda\right)^{\mu}\frac{\Gamma(k+\delta)}{(k-p)!\Gamma(p+\delta)}}{2(k-p)\beta p(p-\gamma)} \right)^{\frac{1}{k-1}}.$$

Each of these results is sharp for the extremal function given by (1.5).

Remark. Other works like the ones in [9-18] can be found by using this derivative operator.

Acknowledgement

The work here is supported by LRGS/TD/2011/UKM/ICT/03/02.

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