

## Fekete -Szego Problem for Starlike of Complex functions of Order $b$ Related to Generalized Derivative Operator

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### الملخص

الغرض من هذه الورقة هو دراسة وحل مشكلة Fekete-Szego لبعض الفئات الفرعية الجديدة لدالة شبيهة بالنجوم و دالة محدبة مرتبات بالعدد المركب  $b$  والتي يرمز لها بالرمز  $s_b^{\alpha, n}(m, q, \lambda)$  و  $c_b^{\alpha, n}(m, q, \lambda)$  على التوالي التي تم الحصول عليها بعامل تفاضلي معمم  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$  المعروف في [8]. مختلف الحالات الخاصة المعروفة أو الجديدة لنتائجنا .

### Abstract

The purpose of the present paper is to investigate the Fekete-Szego problem for certain new subclasses starlike and convex functions of complex of order  $b$  denoted by  $s_b^{\alpha, \delta}(m, q, \lambda)$  and  $c_b^{\alpha, \delta}(m, q, \lambda)$  respectively, which involving certain a generalized derivative operator  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$  defined in [8]. Various known or new special cases of our results are also pointed out.

**Keywords:** Analytic functions; Starlike of complex of order  $b$  ; Convex of complex of order  $b$ , generalized derivative operator

### Introduction and Preliminaries

Fekete-Szegö problem may be considered as one of the most important results about univalent functions, which is related to coefficients of a function's Taylor series and was introduced by Fekete and Szegö [1], studied a special inequality which arises naturally from a combination of two coefficients  $a_2$  and  $a_3$  of a class of univalent analytic normalized function  $S$ . They obtained sharp upper bound of  $|a_3 - \mu a_2^2|$ , where  $\mu$  is real. The problem of maximizing the absolute value of the functional  $a_3 - \mu a_2^2$  is called Fekete-Szegö problem. After 30 years or so, Keogh and Merkes [2] solved the problem for certain subclasses of univalent functions. Then Koepf [3] gave excellent results for the class of close-to-convex functions. Moreover, this functional has also been studied for  $\mu$  as real as well as complex number. And many others follow the

same problems with different techniques for different classes. For other examples defined on various classes can be read in ([3]- [5] and [8]).

Let  $A$  denote the family of analytic functions  $f$  in the unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$  normalized by  $f(0) = 0 = f'(0) - 1$ . If  $f \in A$  then  $f$  has the following representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \quad (1)$$

We also consider  $S$  the class of those functions from  $A$  which are univalent in  $\mathbb{U}$ . Nasr and Aouf [6, 7] introduced  $s^*(b)$  and  $C(b)$ , the class of starlike functions of complex order and the class of convex functions of complex order respectively. More precisely, the function  $f \in A$  is said to be in the class  $s^*(b)$ , if it satisfies the following condition

$$s^*(b) = \operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, z \in \mathbb{U} \text{ and } b \neq 0.$$

Similarly, the function  $f \in A$  is said to be in the class  $C(b)$ , if it satisfies the following condition

$$C(b) = \operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > 0, z \in \mathbb{U} \text{ and } b \neq 0.$$

Observe that  $s^*(1)$  and  $C(1)$  represent standard starlike and convex univalent functions, respectively.

Nagat and Duras in ([8],[9]) have recently introduced generalized derivative operator  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$  as the following:

For the function  $f \in A$  given by (1), we define a new generalized derivative operator as follows:

$$\mathcal{D}^{\alpha, \delta}(m, q, \lambda)f(z) = z + \sum_{k=2}^{\infty} k^{\alpha} \left(1 + \frac{k-1}{1+q} \lambda\right)^m c(\delta, k) a_k z^k,$$

where  $\delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,  $m \in \mathbb{Z}$ ,  $\lambda, q \geq 0$  and  $c(\delta, k) = \frac{(\delta+1)_{k-1}}{(1)_{k-1}}$ ,

where  $(x)_k$  denotes the Pochhammer symbol (or the shifted factorial) defined

by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0 \\ x(x+1)(x+2) \dots (x+k-1) & \text{for } k \in \mathbb{N} = 1, 2, 3, \dots \end{cases}$$

By specializing the parameters of  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$ , we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [10];

$$\mathcal{D}^{0, n}(0, q, \lambda) \equiv \mathcal{D}^{0, n}(1, 0, 0); (n \in \mathbb{N}_0) \equiv R^n = z + \sum_{k=2}^{\infty} c(n, k) a_k z^k.$$

- The derivative operator introduced by Sălăgean [12];

$$\mathcal{D}^{\alpha, 0}(0, q, \lambda) \equiv \mathcal{D}_1^{0, 0}(n, 0, 1); (n \in \mathbb{N}_0) \equiv D^n = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

- The generalized Salagean derivative operator introduced by Oboudi [11];

$$\mathcal{D}^{0, 0}(n, 0, \lambda); (n \in \mathbb{N}_0) \equiv \mathcal{D}_\lambda^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k.$$

- The generalized Ruscheweyh derivative operator introduced by Darus and Al-Shaqsi [13];

$$\mathcal{D}^{0, n}(1, 0, \lambda); (n \in \mathbb{N}_0) \equiv R_\lambda^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1)) c(n, k) a_k z^k.$$

- The derivative operator introduced by Catas [17];

$$\mathcal{D}^{0, \beta}(m, l, \lambda); (m \in \mathbb{N}_0) \equiv \mathcal{D}^m(\lambda, \beta, l) = z + \sum_{k=2}^{\infty} \left( \frac{1 + \lambda(k-1) + l}{1+l} \right)^m c(\beta, k) a_k z^k.$$

- The integral operator introduced by Cho and T. H. Kim [14];

$$\mathcal{D}^{1, 0}(-n, \lambda, 1) \equiv I_n^\lambda = z + \sum_{k=2}^{\infty} k \left( \frac{1 + \lambda}{k + \lambda} \right)^n a_k z^k.$$

Using the operator  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$ , we now introduce the following classes

Definition 1.1. We say that a function  $f \in A$  is in the class  $s_b^{\alpha, \delta}(m, q, \lambda)$  if

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{z \mathcal{D}^{\alpha, \delta}(m, q, \lambda) f'(z)}{\mathcal{D}^{\alpha, \delta}(m, q, \lambda) f(z)} - 1 \right) \right\} > 0, z \in \mathbb{U} \text{ and } b \neq 0$$

By giving specific values to  $m, \alpha, \delta$  and  $b$ , we obtain the following important subclass studied by  $s_{1-b}^{0,n}(0, q, \lambda) = s^*(1-b)$  (Nasr and Aouf [6]).

Definition 1.2. We say that a function  $f \in A$  is in the class  $C_b^{\alpha, \delta}(m, q, \lambda)$  if

$$C(b) = \operatorname{Re} \left\{ 1 + \frac{1}{b} \left( \frac{z \mathcal{D}^{\alpha, \delta}(m, q, \lambda) f''(z)}{\mathcal{D}^{\alpha, \delta}(m, q, \lambda) f'(z)} \right) \right\} > 0, z \in \mathbb{U} \text{ and } b \neq 0$$

Note that  $f \in C_b^{\alpha, \delta}(m, q, \lambda) \Leftrightarrow zf' \in S_b^{\alpha, \delta}(m, q, \lambda)$ .

## 2. Main results

Before we consider how the Taylor series coefficients of functions in the classes  $S_b^{\alpha, \delta}(m, q, \lambda)$  and  $C_b^{\alpha, \delta}(m, q, \lambda)$  might be bounded, let us first consider this problem for the Caratheodory functions.

Let  $P$  be the family of all functions  $p$  analytic in  $\mathbb{U}$  for which  $\operatorname{Re}\{p\} > 0$ ,

given by  $p(z) = 1 + c_1 z + c_2 z^2 + \dots, z \in \mathbb{U}$ .

**Lemma 2.1** ([15]) *If  $p \in P$  then*

$$|c_k| \leq 2, \quad \text{for all } k \in \mathbb{N}.$$

**Lemma 2.2** ([16]) *If  $p_1(z) = 1 + c_1 z + c_2 z^2 + \dots$  is an analytic function with positive real part in  $\mathbb{U}$ , then*

$$|c_2 - \nu c_1^2| \leq 2 \max\{1, |2\nu - 1|\}.$$

The result is sharp for the function

$$p_1(z) = \frac{(1+z)}{(1-z)} \quad \text{or} \quad p_1(z) = \frac{(1+z^2)}{(1-z^2)}.$$

**Theorem 2.3.** Let  $b$  be nonzero complex number. If  $f$  of the form (1) is in  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$ , then

$$|a_2| \leq \frac{(1+q)|b|}{2^{\alpha-1}(1+q+\lambda)^m (\delta+1)}$$

$$|a_3| \leq \frac{2|b|(1+q)}{3^\alpha (1+q+2\lambda)^m (\delta+1)(\delta+2)} \max[1, 1 + |1 + 2b| - 1].$$

**Proof.**

By the definition of the class  $s_b^{\alpha, \delta}(m, q, \lambda)$  there exists  $p \in P$  such, that

$$\frac{z \mathcal{D}^{\alpha, \delta}(m, q, \lambda) f'(z)}{\mathcal{D}^{\alpha, \delta}(m, q, \lambda) f(z)} = 1 - b + b p(z),$$

so that,

$$\frac{z \left[ 1 + 2^{\alpha+1} \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) a_2 z + 3^{\alpha+1} \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^2 + \dots \right]}{z + 2^\alpha \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) a_2 z^2 + 3^\alpha \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^3 + \dots} = 1 - b + b[1 + c_1 z + c_2 z^2 + \dots],$$

which implies the equality

$$\begin{aligned} z + 2^{\alpha+1} \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) a_2 z^2 + 3^{\alpha+1} \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 z^3 + \dots \\ = z + \left[ c_1 b + 2^\alpha \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) \right] a_2 z^2 \\ + \left[ c_2 b + c_1 b 2^\alpha \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) \right. \\ \left. + 3^\alpha \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} \right] a_3 z^3 \\ + \left[ c_3 b + \left( \frac{1+q+\lambda}{1+q} \right)^m c(\delta, 2) c_2 b \right. \\ \left. + c_1 b 3^\alpha \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} \right] a_4 z^4 + \dots \end{aligned}$$

Equating the coefficients of both sides we have

$$2^\alpha \left( \frac{1+q+\lambda}{1+q} \right)^m (\delta+1) a_2 = c_1 b,$$

$$3^\alpha \left( \frac{1+q+2\lambda}{1+q} \right)^m \frac{(\delta+1)(\delta+2)}{2} a_3 = \frac{b}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{(1+2b)}{4} c_1^2 b$$

so

$$a_2 = \frac{bc_1(1+q)}{2^\alpha(1+q+\lambda)^m(\delta+1)}, \quad a_3 = \frac{b(c_2+bc_1^2)(1+q)}{3^\alpha(1+q+\lambda)^m(\delta+1)(\delta+2)} \quad (2)$$

Taking into account (2) and Lemma 2.1, we obtain

$$\begin{aligned} |a_2| &= \left| \frac{bc_1(1+q)}{2^\alpha(1+q+\lambda)^m(\delta+1)} \right| \\ &\leq \frac{2|b|(1+q)}{2^\alpha(1+q+\lambda)^m(\delta+1)} \\ &= \frac{|b|(1+q)}{2^{\alpha-1}(1+q+\lambda)^m(\delta+1)} \\ |a_3| &= \left| \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} + bc_1^2 \right] \right| \\ &\leq \frac{|b|(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ 2 - \frac{|c_1|^2}{2} + \frac{|1+2b|}{2}|c_1|^2 \right] \\ &= \frac{|b|(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ 2 + |c_1|^2 \frac{|1+2b|-1}{2} \right] \\ &= \frac{2|b|(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \max[1, 1 + |1+2b|-1]. \end{aligned}$$

First, we consider the case, when  $|a_3 - \mu a_2^2|$  for complex  $\mu$ .

**Theorem 2.4.** Let  $b$  be a nonzero complex number and let  $s_b^{\alpha,\delta}(m,q,\lambda)$ . Then for  $\mu$  complex then

$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \max \left[ 1, \left| 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right| \right]$$

For each  $\_$  there is a function in  $s_b^{\alpha,\delta}(m,q,\lambda)$  such that equality holds.

**Proof.** Applying (2) we have

$$\begin{aligned}
 a_3 - \mu a_2^2 &= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} [c_2 + bc_1^2] \\
 &\quad - \frac{\mu b^2 c_1^2 (1+q)^2}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)^2} \\
 &= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ c_2 + bc_1^2 \right. \\
 &\quad \left. - \frac{\mu b c_1^2 (1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right] \\
 &= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ c_2 + \frac{2bc_1^2}{2} - \frac{c_1^2}{2} + \frac{c_1^2}{2} \right. \\
 &\quad \left. - \frac{2\mu b c_1^2 (1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2 \cdot 2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right] \\
 &= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ c_2 - \frac{c_1^2}{2} \right. \\
 &\quad \left. + \frac{c_1^2}{2} \left( 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right) \right], \quad (3)
 \end{aligned}$$

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{aligned}
 &\frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ 2 - \frac{|c_1^2|}{2} \right. \\
 &\quad \left. + \frac{|c_1^2|}{2} \left| 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right| \right] \\
 &= \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ 2 + \frac{|c_1^2|}{2} \left| 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right| - 1 \right].
 \end{aligned}$$

Then, with the aid of Lemma 2.2, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \max \left[ 1, \left| 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right| \right]$$

Taking  $\delta = q = m = \alpha = 0$  and  $b = 1$  in Theorem 2.4, we have

**Corollary 2.5** [2] If  $f \in s^*$ , then for  $\mu \in \mathbb{C}$  we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.$$

Moreover, for each  $\mu$ , there is a function in  $S$  such that equality holds.

**Theorem 2.6** Let  $b > 0$ , and let  $s_b^{\alpha, \delta}(m, q, \lambda)$ . Then for  $\mu$  real,

Then,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2b(1+q)}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} \left( 1 + 2b \left( 1 - \mu \frac{3^\alpha \mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)(1+q+\lambda)^{2m}} \right) \right) & \text{if } \mu \leq \sigma_1, \\ \frac{2b(1+q)}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{2b(1+q)}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} \left( -1 - 2b \left( 1 + \mu \frac{3^\alpha \mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)(1+q+\lambda)^{2m}} \right) \right) & \text{if } \mu \geq \sigma_2. \end{cases}$$

Where

$$\sigma_1 = \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}}{3^\alpha(1+q)^m(1+q+2\lambda)^m(\delta+2)},$$

$$\sigma_2 = \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[1+b]}{3^\alpha b(1+q)^m(1+q+2\lambda)^m(\delta+2)}.$$

Moreover for each  $\mu$ , there is a function in  $s_b^{\alpha, \delta}(m, q, \lambda)$  such that equality hold

**Proof** By (3), we obtain

$$a_3 - \mu a_2^2 = \frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2} \left( 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right) \right]$$

First, let.  $\mu \leq \sigma_1$ , In this case, by (3), Lemma 2.1 and give

$$|a_3 - \mu a_2^2| \leq$$

$$\frac{b(1+q)}{3^\alpha(1+q+2\lambda)^m(\delta+1)(\delta+2)} \left[ 2 - \frac{|c_1^2|}{2} + \frac{|c_1^2|}{2} \left| 1 + 2b - \frac{2\mu b(1+q)(\delta+2)3^\alpha(1+q+2\lambda)^m}{2^{2\alpha}(1+q+\lambda)^{2m}(\delta+1)} \right| \right]$$

$$\leq \frac{2b(1+q)}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} \left( 1 + 2b \left( 1 - \mu \frac{3^\alpha \mu(\delta+2)(1+q+2\lambda)^m(1+q)}{2^{2\alpha}(\delta+1)(1+q+\lambda)^{2m}} \right) \right)$$

Now let  $\sigma_1 \leq \mu \leq \sigma_2$ , Then, using the above calculations, we get



$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^\alpha (\delta+1)(\delta+2)(1+q+2\lambda)^m}$$

Finally, if  $\mu \geq \sigma_2$ , then we obtain

$$|a_3 - \mu a_2^2| \leq \frac{b(1+q)}{3^\alpha (1+q+2\lambda)^m (\delta+1)(\delta+2)} \left[ 2 - \frac{|c_1^2|}{2} + \frac{|c_1^2|}{2} \left| -1 - 2b + \frac{2\mu b(1+q)(\delta+2)3^\alpha (1+q+2\lambda)^m}{2^{2\alpha} (1+q+\lambda)^{2m} (\delta+1)} \right| \right] \leq \frac{2b(1+q)}{3^\alpha (\delta+1)(\delta+2)(1+q+2\lambda)^m} \left( -1 - 2b \left( 1 + \mu \frac{3^\alpha \mu (\delta+2)(1+q+2\lambda)^m (1+q)}{2^{2\alpha} (\delta+1) (1+q+\lambda)^{2m}} \right) \right).$$

Using the well-known Alexander relation, we easily obtain bounds of coefficients and a solution of the Fekete-Szegő problem for the class  $c_b^{\alpha, \delta}(m, q, \lambda)$

**Theorem 2.7.** Let  $b$  be nonzero complex number. If  $f$  of the form (1) is in  $\mathcal{D}^{\alpha, \delta}(m, q, \lambda)$ , then

$$|a_2| \leq \frac{(1+q)|b|}{2^\alpha (1+q+\lambda)^m (\delta+1)}$$

$$|a_3| \leq \frac{2(1+q)|b|}{3^{\alpha+1} (1+q+2\lambda)^m (\delta+1)(\delta+2)} [1 + |1 + 2b|]$$

$$|a_3 - \mu a_2^2| \leq \frac{2b(1+q)}{3^{\alpha+1} (1+q+2\lambda)^m (\delta+1)(\delta+2)} \max \left[ 1, \left| 1 + 2b - \frac{3\mu b(1+q)(\delta+2)3^\alpha (1+q+2\lambda)^m}{2^{2\alpha+1} (1+q+\lambda)^{2m} (\delta+1)} \right| \right].$$

Taking  $\delta = q = m = \alpha = 0$  and  $b = 1$  in Theorem 2.7, we have

**Corollary 2.8** [2] *If  $f \in s^*$ , then for  $\mu \in \mathbb{C}$  we have*

$$|a_3 - \mu a_2^2| \leq \max \left\{ \frac{1}{3}, |\mu - 1| \right\}.$$

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## References

- [1] M. Fekete and G. Szego, Eine Bemerkung uber ungerade schlichte funktionen, J. London Math. Soc. 8 (1933), 85-89.
- [2] F.R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic function, Proc. Amer. Math. Soc. 20 (1969), 8-12.
- [3] W. Koeph, On the Fekete-Szego problem for close-to-convex functions, Proc Amer. Math. Soc. 101 (1987), 89-95
- [4] M.K. Aouf and F.M. Abdulkarem, Fekete {Szego inequalities for certain class of analytic functions of complex order, International Journal of Open Problems in Complex Analysis 6 (1) (2014),1-13.
- [5] M. Darus, D.K. Thomas, On the Fekete\_Szegö theorem for close-to-convex functions, Math. Japonica 44 (1996) 507\_511.
- [6] M.A. Nasr, M.K. Aouf, Starlike function of complex order, J. Natur. Sci. Math. 25 (1985) 1\_12.
- [7] M.A. Nasr, M.K. Aouf, On convex functions of complex order, Mansoura Sci. Bull. (1982) 565\_582.
- [8] Nagat.M. Mustafa, and Maslina Darus, The Fekete-Szego problem for starlike functions of order associated with generalized derivative operator.AIP Conf. Proc. (2012) 15(22): 938-944.
- [9] Nagat.m.Mustafa, and maslina Darus, Some extensions of sufficient conditions for univalence of an integral operator.Journal of Concrete and Applicable Mathematics . Apr2013, Vol. 11 Issue 2, p160-167.
- [10] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math.Soc.* 49(1975), 109-115.
- [11] F.M. AL-Oboudi, On univalent functions defined by a generalized Operator Salagean, Int. J. Math. Math. Sci, 27 (2004),1429-1436.

[12] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in Math Springer-Verlag, **1013**,(1983), 362 Moreover for each  $\mu$ , there is a function in  $S$  such that equality holds

[13] M. Darus and K. Al-Shaqsi, Differential Subordination with generalised derivative operator, Int.j.comp Math. Sci **2(2)**(2008),75 -78.

[14] N. E. Cho and T. H. Kim, Multiplier transformations and strongly close to-convex functions, Bull. Korean Math. Soc,40 (2003), 399-410.

[15] Duren, P.L.: Univalent Functions, Grundlehren der Mathematics. Wissenschaften, Bd., p. 259. Springer, NewYork (1983)

[16] W. Ma and D. Minda, A unified treatment of some special classes of univalent functions ,in: Proceedings of the Conference on Complex Analysis, Z. Li, F. Ren,L. Yang, and S. Zhang(Eds.) Internat. Press (1994), 157-169.

[17] A. Catas, On a Certain Differential Sandwich Theorem Associated with a New Generalized Derivative Operator, General Mathematics. 4 (2009), 83-95.