Int. J. Open Problems Complex Analysis, Vol. 3, No. 1, March 2011
ISSN 2074-2827; Copyright © ICSRS Publication, 2011
www.i-csrs.org

Coefficient Estimates for Certain Class of Analytic Functions

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Abstract

In this paper, we consider some coefficient estimates for certain class of analytic functions denoted by $CL(\beta)$***, and satisfying*** $|K^2(f,z) - (1-\beta)| < 1-\beta, \quad (\frac{1}{2} < \beta < 1)$.

Keywords: Analytic functions, convex functions, k-starlike functions, strongly starlike functions, subordination.

AMS Mathematics Subject Classification (2000): 30C45.

1 Introduction

Let H denote the class of analytic functions which are in the open unit disc $U = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let A denote the subclass of H consisting of functions normalized by $f(0) = 0, f'(0) = 1$.

Let functions f and g be analytic in U. Then a function f is said to be subordinate to a function g, if there exists a function w, analytic in U such that $w(0) = 0, |w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. We denote this subordination by $(f \prec g)$. In particular, if a function g is univalent in U we have the following equivalence

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Robertson introduced in [1] the classes $S^*(\beta)$,$C^*(\beta)$ of starlike and convex functions of order $\beta \leq 1$, in U which is defined by:

$$S^*(\beta) = Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \quad (z \in U),$$

and

$$C^*(\beta) = Re\left\{\frac{zf''(z)}{f(z)} + 1\right\} > \beta \quad (z \in U).$$

If $(0 \leq \beta < 1)$, then a function in either of this set is univalent. If $\beta < 0$, then it may fail to be univalent.

In this way, many interesting classes of analytic functions have been studied such as the class SL^* which is defined by

$$SL^* = \{f \in A : \left|\frac{zf'(z)}{f(z)} - 1\right| < 1\}, \quad z \in U.$$

and can be found in [5].

And also $SL^* \subset SS^*(\frac{1}{2}) \subset S^*$, is the class $SS^*(\beta)$ of strongly starlike functions of order β and given by

$$SS^*(\beta) = \left\{f \in A : |Arg\frac{zf'(z)}{f(z)}| < \frac{\beta\pi}{2}\right\}, \quad (0 < \beta \leq 1),$$

which was studied in [2].

Moreover, $k-ST \subset SL^*$, for $k \geq 2+\sqrt{2}$, where $k-ST$ is the class of k-starlike functions introduced in [3] such that

$$k - ST := \left\{f \in A : Re\left|\frac{zf'(z)}{f(z)}\right| > k\left|\frac{zf'(z)}{f(z)} - 1\right|\right\}, \quad k \geq 0.$$

Let us denote

$$K(f, z) := \frac{zf''(z)}{f'(z)}.$$

In this paper we consider the class $CL(\beta)$, given by

$$CL(\beta) := \left\{f \in A : |K^2(f, z) - (1 - \beta)| < 1 - \beta\right\}. \tag{1.1}$$

Notice that, $L := \{w \in \mathbb{C}, \Re(w) > 0 : |w^2 - (1 - \beta)| < 1 - \beta\}$ is the interior of the right half of the lemniscate of Bernoulli $\gamma : (x^2+y^2)^2-2(1-\beta)(x^2-y^2) = 0$. It can be verified that$\{w : |w^2 - 1| < 1\} \subset L, \quad for(\frac{1}{2} < \beta < 1)$, thus $SL^* \subset CL(\beta)$.

Also, it is easy to see $L \subset \left\{w : |Argw| < \frac{\beta\pi}{2}\right\}$, thus $CL(\beta) \subset SS^*(\beta)$.

Corollary 1.1 *If* $k \geq 2+\sqrt{2}, \quad \frac{1}{2}<\beta<1,$ *then*

$$k-ST \subset CL(\beta) \subset SS^*(\beta) \subset S^*.$$

Theorem 1.2 *The function f belongs to the class $CL(\beta)$ if and only if there exists an analytic function*

$$q \in H, q(z) \prec q_0(z)=\sqrt{(1-\beta)z+(1-\beta)}, q_0(0)=\sqrt{1-\beta},$$

such that

$$f(z)=\int_0^z y \exp \int_0^y \frac{q(t)-1}{t}\, dtdy. \tag{1.2}$$

Let

$$q_1(z)=\frac{3+2z}{3+z}, \quad q_2(z)=\frac{5+3z}{5+z}, \quad q_3(z)=\frac{8+4z}{8+z},$$

$$q_4(z)=\frac{8+4z}{8+z} \quad \textit{and} \quad q_5(z)=\frac{9+5z}{9+z}.$$

Now, since $q_i(z) \prec q_0(z)$for $i=1,2,3,4,$ we have then by (1.2) that

$$f_1(z)=\frac{z^2}{2}+\frac{z^3}{9}, \quad f_2(z)=\frac{1}{2}z^2+\frac{2}{15}z^3+\frac{z^4}{100}, \quad f_3(z)=\frac{1}{2}z^2+\frac{2}{8}z^3+\frac{1}{128}z^4+\frac{1}{2560}z^5,$$

$$f_4(z)=\frac{1}{2}z^2+\frac{4}{27}z^3+\frac{1}{54}z^4+\frac{4}{3645}z^5+\frac{1}{39366}z^6.$$

2 Main results

Theorem 2.1 *If a function $f(z)=z+a_2z^2+...+a_nz^n$ belongs to the class $CL(\beta)$, then*

$$\sum_{k=2}^{\infty} k^2(k-1-2(1-\beta))|a_k|^2 \leq 2(1-\beta), \quad (\frac{1}{2}<\beta<1). \tag{2.1}$$

Proof. If a function $f \in CL(\beta)$, then $K(f,z) \prec q_0(z)=\sqrt{(1-\beta)z+(1-\beta)}$. Thus we can write $K(f,z)=\sqrt{(1-\beta)w(z)+(1-\beta)}$, where w satisfies $w(0)=0, |w(z)|<1 \quad \text{for } |z|<1,$

and

$$K(f,z)=\frac{zf''(z)}{f'(z)}.$$

Whence

$$(1-\beta)(f'(z))^2 = (zf''(z))^2 - (1-\beta)w(z)(f'(z))^2.$$

From this we can obtain

$$2(1-\beta)\pi\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k-2} = (1-\beta)\int_0^{2\pi} \left|f'(re^{i\theta})\right|^2 d\theta$$

$$\geq (1-\beta)\int_0^{2\pi} |w(re^{i\theta})|\left|f'(re^{i\theta})\right|^2 d\theta$$

$$= \int_0^{2\pi} \left|(re^{\theta}f''(re^{i\theta})^2 - (1-\beta)f'(re^{i\theta})^2\right| d\theta$$

$$4(1-\beta)\pi\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k-2} \geq 2\pi\sum_{k=1}^{\infty} k^2(k-1)|a_k|^2 r^{2k-2},$$

for $0 < r < 1$,

and derive into the following inequalities

$$2(1-\beta)\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k-2} \geq \sum_{k=1}^{\infty} k^2(k-1)|a_k|^2 r^{2k-2}$$

$$2(1-\beta)\sum_{k=1}^{\infty} k^2|a_k|^2 r^{2k-2} - \sum_{k=1}^{\infty} k^2(k-1)|a_k|^2 r^{2k-2} \geq 0.$$

Eventually, if we let $r \to 1^-$, then we obtain

$$\sum_{k=2}^{\infty} k^2(k-1-2(1-\beta))|a_k|^2 \leq 2(1-\beta).$$

Corollary 2.2 *If a function* $f(z) = z + a_2z^2 + a_3z^3 + ... + a_nz^n$ *belongs to the class* $CL(\beta)$, *then*

$$|a_k| \leq \sqrt{\frac{2(1-\beta)}{k^2(k-1-2(1-\beta))}} \qquad for \qquad k \geq 2 \quad and \quad \frac{1}{2} < \beta < 1.$$

Theorem 2.3 *If a function* $f(z) = \sum_{k=1}^{\infty} a_kz^k$ *belongs to the class* $CL(\beta)$, *then*

$$|a_2| \leq \frac{1}{4} \quad , \quad |a_3| \leq \frac{1}{6} \quad , \quad |a_4| \leq \frac{1}{8}. \tag{2.2}$$

These results are sharp.

Proof: If $f(z)=\sum_{k=1}^{\infty}a_kz^k$ belongs to the class $CL(\beta)$, then $(1-\beta)(f'(z))^2 = (zf''(z))^2-(1-\beta)w(z)(f'(z))^2$, where w satisfies $w(0)=0, |w(z)|<1 \quad for \quad |z|<1$.

Let us denote

$$(zf''(z))^2=\sum_{k=2}^{\infty}A_kz^k, \qquad f'^2(z)=\sum_{k=0}^{\infty}B_kz^k, \qquad w(z)=\sum_{k=1}^{\infty}C_kz^k.$$

Then we have

$$A_k = 4(a_2)^2z^2+24a_2a_3z^3+(48a_2a_4+36a_3{}^2)z^4+(80a_2a_5+144a_3a_2)z^5+\dots, \tag{2.3}$$

and

$$B_k = a_1z^2+2a_2z^3+(2a_3+a_2)z^4+(2a_2a_4-a_3{}^2-a_1a_5)z^6+\dots, \tag{2.4}$$

such that

$$\sum_{k=2}^{\infty}A_kz^k-(1-\beta)\sum_{k=0}^{\infty}B_kz^k=\sum_{k=1}^{\infty}C_kz^k\sum_{k=0}^{\infty}B_kz^k. \tag{2.5}$$

Thus we obtain

$$A_2=4a_2,\ A_3=24a_2a_3,\ A_4=48a_2a_4+12a_3{}^2,\ A_5=80a_2a_5+144a_3a_4, \tag{2.6}$$

and

$$B_0=a_1{}^2=1,\ B_1=4a_1a_2=4a_2,\ B_2=6a_3a_1+4^2,\ B_3=8a_1a_4+12a_2a_3. \tag{2.7}$$

Equating second and third coefficients of (2.5), we obtain

(I) $-(1-\beta)B_1=(1-\beta)B_0C_1$.

(II) $A_2-(1-\beta)B_2=(1-\beta)(C_2B_0+C_1B_1)$.

(III) $A_3-(1-\beta)B_3=(1-\beta)(C_3B_0+C_2B_1+C_1B_2)$.

And so by (2.5), (2.6) and (2.7), we have

$$a_2=\frac{1}{4}C_1, \qquad a_3=-\frac{1}{6}C_2+\frac{1}{6}C_1^2,$$

$$a_4=\frac{1}{8}C_3+\frac{(3-2\beta}{24(1-\beta)}C_1C_2-\frac{1}{32}C_1^2+\frac{(2-\beta}{16(1-\beta)}C_1^3$$

Using a well-known inequality $|C_k|\le 1, \sum_{k=1}^{\infty}|C_k|^2\le 1$, we obtain (2.2).

Other work related to coefficient problems for different class of analytic functions can be found in ([6],[7],[8]).

3 Open Problem

The method here is employed from the work done by Sokol [4]. It is interesting to see other results in a similar technique for different classes such as the starlike subclasses with respect to symmetric points. We did try, but failed to get any results and it is left for the readers to tackle this problem.

Acknowledgement: The work here was supported by UKM-ST-06-FRGS0244-2010.

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