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SOME PROPERTIES OF CERTAIN SUBCLASSES OF p -VALENT FUNCTIONS DEFINED BY A LINEAR DERIVATIVE OPERATOR

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ABSTRACT. In the present paper, we introduce new classes of p -valent functions defined by using a generalized linear derivative operator with negative coefficients in the unit disk. The results presented here include coefficient estimates, extreme points and distortion properties for the aforementioned classes.

Key words. p -valent functions, starlike, convex, distortion theorems, linear derivative operator.

AMS Mathematics Subject Classification (2000): 30C45.

1. DEFINITION AND PRELIMINARIES

Let A_p denote the class of functions of the form :

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}). \quad (1.1)$$

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which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in A_p$ is called p -valent starlike of order β and type γ , if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\gamma} \right| < \beta, \quad (1.2)$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $S^*(p, \gamma, \beta)$ the class of p -valent starlike functions of order γ and type β . A function $f \in A_p$ is called p -valent convex functions of order β and type γ , if it satisfies

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{1 + \frac{zf''(z)}{f'(z)} + p - 2\gamma} \right| < \beta, \quad (1.3)$$

where $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$. We denote by $K(p, \gamma, \beta)$ the class of p -valent convex functions of order γ and type β .

From (1.2) and (1.3), we note that: $f(z) \in K(p, \gamma, \beta)$ if, and only if,

$$\frac{zf'}{p} \in S^*(p, \gamma, \beta).$$

The classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ were considered by Aouf [2] and Hossen [3]. For $\beta = 1$, reduced to the class $S^*(p, \gamma, 1) = S^*(p, \gamma)$ which was studied by Patil and Thakare [4], and the class $K(p, \gamma, 1) = K(p, \gamma)$ given by Owa [5].

Let T_p denote the subclass of A_p consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N}). \quad (1.4)$$

We denote by $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$, the classes obtained by taking intersections, respectively, of the classes $S^*(p, \gamma, \beta)$ and $K(p, \gamma, \beta)$ with the class T_p . Thus we have

$$T^*(p, \gamma, \beta) = S^*(p, \gamma, \beta) \cap T_p,$$

and

$$C(p, \gamma, \beta) = K(p, \gamma, \beta) \cap T_p.$$

The classes $T^*(p, \gamma, \beta)$ and $C(p, \gamma, \beta)$ were studied by Aouf [2] and Hossen [3]. In particular, the classes $T^*(p, \gamma, 1) = T^*(p, \gamma)$ and $C(p, \gamma, 1) = C(p, \gamma)$ were introduced by Owa [5]. Also the classes $T^*(1, \gamma, 1) = T^*(\gamma)$ and $C(1, \gamma, 1) = C(\gamma)$ were studied by Silverman [6].

For functions $f \in A_p$, given by (1.1), and g given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \quad (p \in \mathbb{N}),$$

the Hadamard product (or convolution) of functions f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z), \quad (p \in \mathbb{N}).$$

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(x)_k =$

$$\begin{cases} 1 & \text{for } k = 0, \\ x(x+1)(x+2)\dots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \dots\}. \end{cases}$$

The authors in [1] have recently introduced a new generalized linear derivative operator $D_p^{\alpha, \delta}(\mu, q, \gamma)$, as the following:

Definition 1.1. For $f \in A_p$, the linear operator $D_p^{\alpha, \delta}(\mu, q, \gamma)$ is defined by $D_p^{\alpha, \delta}(\mu, q, \gamma) : A_p \rightarrow A_p$ as:

$$D_p^{\alpha, \delta}(\mu, q, \gamma)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^{\alpha} \left(1 + \frac{k-p}{p+q}\lambda\right)^{\mu} c(\delta, k) a_k z^k, \quad (1.5)$$

where $\lambda, \mu, q \geq 0$, $k, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$,

and $c(\delta, k) = z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+\delta)}{(k)\Gamma(p+\delta)} z^k$.

Next we define the following new subclasses of p -valent functions as follows:

Definition 1.2. Let $f \in T_p$ be given by (1.4). Then f is said to be in the class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$ if, and only if,

$$\left| \frac{\frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} - p}{\frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} + p - 2\gamma} \right| < \beta,$$

where $D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)$ is given by (1.5) and $\lambda, \mu, q \geq 0$, $k, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $0 \leq \gamma < p$, $0 < \beta \leq 1$ and $p \in \mathbb{N}$.

Further, a function $f \in T_p$ is said to be in the class $C_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$ if, and only if,

$$\frac{zf'}{p} \in T_p^{\alpha,\delta}(\mu, q, \gamma, \beta).$$

We note that, by specializing the parameters $\alpha, \delta, \mu, \lambda, \beta$ and p , we shall obtain the following subclasses which were studied by various authors:

1. For $\alpha = \delta = \mu = 0$ we get $T_p^{0,0}(0, q, \gamma, \beta) = T^*(p, \gamma, \beta)$, is the class of p -valent starlike function of order γ and type β which was studied by Aouf [2] and Hossen [3].
2. For $\alpha = \delta = \mu = 0$ and $p = 1$, we have $T_1^{0,0}(0, q, \gamma, \beta) = S^*(\gamma, \beta)$, is the class of starlike function of order γ and type β which was studied by Gupta and Jain [7].
3. For $\alpha = \delta = \mu = 0$ and $\beta = 1$, we obtain the class $T_p^{0,0}(0, q, \gamma, 1) = T^*(p, \gamma)$, which was introduced by Owa [5].
4. For $\alpha = \delta = \mu = 0$, $p = 1$ and $\beta = 1$ we obtain the class $T_1^{0,0}(0, q, \gamma, 1) = T^*(\gamma)$, which was studied by Silverman [6].
5. For $\alpha = \delta = q = 0, \mu = 1$ and $p = 1$, we have the class $C_1^{0,0}(1, 0, \gamma, \beta) = C^*(\gamma, \beta)$, which was studied by Gupta and Jain [7].
6. For $\alpha = \delta = q = 0, \mu = 1$, we have the class $C_p^{0,0}(1, 0, \gamma, \beta) = C(p, \gamma, \beta)$, is the class of p -valent convex function of order γ and type β , studied by Aouf [2] and Hossen [3].
7. For $\alpha = \delta = q = 0, \mu = 1$, and $\beta = 1$, we have the class $C_p^{0,0}(1, 0, \gamma, 1) = C(p, \gamma)$, studied by Owa [5].

8. For $\alpha = \delta = q = 0, \mu = 1, \beta = 1$, and $p = 1$, we obtain the class $C_1^{0,0}(1, 0, \gamma, 1) = C(\gamma)$, studied by Silverman [6].

2. COEFFICIENT ESTIMATES

Theorem 2.1. *A function f belongs to the class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$ if, and only if,*

$$\sum_{k=p+1}^{\infty} \left(((k-p) + \beta(k+p-2\gamma)) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k \right) \leq 2\beta(p-\gamma). \quad (2.1)$$

Proof: Let the function f be in the class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$. Then we have

$$\left| \frac{\frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} - p}{\frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)} + p - 2\gamma} \right| = \left| \frac{\frac{pz^p - \sum_{k=p+1}^{\infty} (k) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k}{z^p - \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k} - p}{\frac{pz^p - \sum_{k=p+1}^{\infty} (k) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k}{z^p - \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k} + p - 2\gamma} \right| \leq \beta.$$

Since $|Re(z)| \leq |z|$ for all z , we have

$$\Re \left\{ \frac{\sum_{k=p+1}^{\infty} (k-p) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k}{-\sum_{k=p+1}^{\infty} (k+p-2\gamma) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k + (2p-2\gamma)} \right\} \leq \beta.$$

Choosing values of z on the real axis, so that $\frac{z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z)}{D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)}$ is real, and letting $z \rightarrow 1^-$, through real axis, we get

$$\begin{aligned} & \sum_{k=p+1}^{\infty} (k-p) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)} a_k z^k \leq \\ & -\beta \left(\sum_{k=p+1}^{\infty} (k+p-2\gamma) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k + \beta(2p-2\gamma) \right), \end{aligned}$$

which implies the assertion (2.1). Conversely, let the inequality (2.1) holds true, then

$$\begin{aligned} & \left| z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z) - p(D_p^{\alpha,\delta}(\mu, q, \gamma)f(z)) \right| - \beta \\ & \left| z(D_p^{\alpha,\delta}(\mu, q, \gamma)f)'(z) + (p-2\gamma)D_p^{\alpha,\delta}(\mu, q, \gamma)f(z) \right|, \end{aligned}$$

$$\sum_{k=p+1}^{\infty} \left(((k-p) + \beta(k+p-2\gamma)) \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} \right) - \beta(2p-2\gamma) \leq 0,$$

by the assumption. This implies that $f \in T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$.

Corollary 2.1. *Let the function f be in the class $T_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$, then*

$$a_k \leq \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}. \quad (2.2)$$

The result (2.2) is sharp for the function f of the form

$$f(z) = z^p - \frac{2\beta(p - \gamma)}{((k - p) + \beta(k + p - 2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} z^k. \quad (2.3)$$

By using the same arguments as in the proof of Theorem 2.1, we can establish the next theorem.

Theorem 2.2. *A function f belongs to the subclass $C_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$, if, and only if,*

$$\sum_{k=p+1}^{\infty} \left(k[(k - p) + \beta(k + p - 2\gamma)] \left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k + \delta)}{k!\Gamma(p + \delta)} a_k z^k \right) \leq 2\beta p(p - \gamma),$$

Corollary 2.2. *Let the function f be in the class $C_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$. Then*

$$a_k \leq \frac{2\beta p(p - \gamma)}{k[(k - p) + \beta(k + p - 2\gamma)]\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}},$$

with equality only for functions of the form

$$f(z) = z^p - \frac{2\beta p(p - \gamma)}{k[(k - p) + \beta(k + p - 2\gamma)]\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} z^k.$$

3. DISTORTION PROPERTIES

In this section, we obtain distortion bounds for the classes $T_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$ and $C_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$.

Theorem 3.1. *If $f \in T_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$, then*

$$|f(z)| \geq r^p - \frac{2\beta(p - \gamma)}{(1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1} \quad (3.1)$$

$$\leq r^p + \frac{2\beta(p - \gamma)}{(1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1}, \quad (3.2)$$

and

$$|f'(z)| \geq pr^{p-1} - \frac{2\beta(p - \gamma)(p + 1)}{(1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^p \quad (3.3)$$

$$\leq p|z|^{p-1} + \frac{2\beta(p - \gamma)(p + 1)}{(1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^p, \quad (3.4)$$

for $z \in \mathbb{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof: Since $f \in T_p^{\alpha, \delta}(\mu, q, \gamma, \beta)$, and in view of inequality (2.1) of Theorem 2.1, we have

$$\begin{aligned} & (1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)} \sum_{k=p+1}^{\infty} a_k \leq \\ & \sum_{k=p+1}^{\infty} \left(((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{k-p}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)} a_k z^k \right) \leq 2\beta(p - \gamma), \end{aligned}$$

or

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{2\beta(p - \gamma)}{(1 + \beta(1 + 2p - 2\gamma))\left(\frac{p+1}{p}\right)^\alpha \left(1 + \frac{\lambda}{p+q}\right)^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}. \quad (3.5)$$

Since

$$r^p - r^{p+1} \sum_{k=p+1}^{\infty} a_k \leq |f(z)| \leq r^p + r^{p+1} \sum_{k=p+1}^{\infty} a_k, \quad (3.6)$$

on using (3.5) and (3.6), we easily arrive at the desired results of (3.2) and (3.1). Furthermore, we observe that

$$pr^{p-1} - (p+1)r^p \sum_{k=p+1}^{\infty} a_k \leq |f'(z)| \leq pr^{p-1} + (p+1)r^p \sum_{k=p+1}^{\infty} a_k, \quad (3.7)$$

On using (3.5) and (3.7), we easily arrive at the desired results of (3.3) and (3.4). Finally, we can see that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp for the function,

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{(1+(1+2p-2\gamma))(1+\frac{\lambda}{p+q})^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}.$$

Similarly, we can prove the following theorem.

Theorem 3.2. *If $f \in C_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$, then*

$$\begin{aligned} |f(z)| &\geq r^p - \frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^\alpha(1+\frac{\lambda}{p+q})^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1} \\ &\leq r^p + \frac{2\beta p(p-\gamma)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^\alpha(1+\frac{\lambda}{p+q})^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^{p+1}, \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq pr^{p-1} - \frac{2\beta p(p-\gamma)(p+1)}{[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^\alpha(1+\frac{\lambda}{p+q})^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^p \\ &\leq pr^{p-1} + \frac{2\beta p(p-\gamma)(p+1)}{(p+1)[1+\beta(1+2p-2\gamma)](\frac{p+1}{p})^\alpha(1+\frac{\lambda}{p+q})^\mu \frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}} r^p, \end{aligned}$$

for $z \in \mathbb{U}$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

4. EXTREME POINTS

Theorem 4.1. *Let $f_p(z) = z^p$ and,*

$$f_k(z) = z^p - \frac{2\beta(p-\gamma)}{((k-p)+\beta(k+p-2\gamma))(\frac{k}{p})^\alpha(1+\frac{(k-p)\lambda}{p+q})^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} z^k.$$

Then f is in the class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$, if, and only if, it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1. \quad (4.1)$$

Proof: Let $f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z)$

$$f(z) = z^p - \frac{2\beta(p-\gamma)}{((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k z^k.$$

Then, in view of (4.1), it follows that

$$\sum_{k=p+1}^{\infty} \frac{((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}{2\beta(p-\gamma)} \times$$

$$\left\{ \frac{2\beta(p-\gamma)}{((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}} \omega_k \right\} = \sum_{k=1}^{\infty} \omega_k = 1 - \omega_1 \leq 1.$$

Thus $f \in T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$.

Conversely, assume that a function f defined by (1.4) belongs to class $T_p^{\alpha,\delta}(\mu, q, \gamma, \beta)$.

Then

$$a_k \leq \frac{2\beta(p-\gamma)}{((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}.$$

We set

$$\omega_k = \frac{((k-p) + \beta(k+p-2\gamma))\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{k!\Gamma(p+\delta)}}{2\beta(p-\gamma)},$$

and $\omega_k = 1 - \sum_{k=1}^{\infty} \omega_k$. Then we have $f(z) = \sum_{k=1}^{\infty} \omega_k f_k(z)$, and hence completes the proof.

Similarly, we can prove the following result:

Theorem 4.2. Let $f_p(z) = z^p$ and,

$$f_k(z) = z^p - \frac{2\beta p(p-\gamma)}{k[(k-p) + \beta(k+p-2\gamma)]\left(\frac{k}{p}\right)^\alpha \left(1 + \frac{(k-p)}{p+q}\lambda\right)^\mu \frac{\Gamma(k+\delta)}{(k)!\Gamma(p+\delta)}} z^k.$$

Then f is in the class $C_p^{\alpha, \delta}(\mu, q, \gamma, \beta, \cdot)$, if, and only if, it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} \omega_k f_k(z),$$

where

$$\omega_k \geq 0, \sum_{k=0}^{\infty} \omega_k = 1.$$

Many other work on p -valent functions related to derivative operator and integral operator can be read in [8]-[10] and [11], respectively.

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