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ON THE FEKETE-SZEGÖ PROBLEM FOR SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY LINEAR DERIVATIVE OPERATOR

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Abstract

In applying a generalized linear derivative operator $D^{\alpha,\delta}(m, q, \lambda)$, a new subclass of analytic functions denoted by $S^{\alpha,\delta}(m, q, \lambda, \phi)$, is introduced. For this class, sharp bounds for the Fekete- Szegö functional $|a_3 - \mu a_2^2|$ are obtained. Also we give applications of our results to certain functions defined through convolution (or Hadamard product) and in particular, we consider a class of functions defined by fractional derivatives. The aim of this paper is to generalize the Fekete-Szegö inequalities given by Srivastava and Mishra [5].

1 Introduction

Let A denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let S be the subclass of A consisting of univalent functions. For two analytic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, their convolution (or Hadamard product) is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

In order to derive our new linear derivative operator, we define the analytic function

$$\varphi^{\alpha}(m,q,\lambda)(z) = z + \sum_{k=2}^{\infty} k^{\alpha}(1 + \frac{k-1}{1+q}\lambda)^{m} z^{k},$$

where $\lambda, m, q \in \mathbb{R}$, $\lambda, m, q \geq 0$.

Now,we introduce the new linear derivative operator as the following:

Definition 1.1 *For $f \in A$ the linear operator $D^{\alpha,\delta}(m,q,\lambda)$ is defined by $D^{\alpha,\delta}(m,q,\lambda) : A \to A$ as*

$$D^{\alpha,\delta}(m,q,\lambda) = \varphi^{\alpha}(m,q,\lambda) * R^{\delta}, \qquad (z \in \mathbb{U}), \tag{1.2}$$

where $k, \delta \in \mathbb{N}_0 = \{0, 1, 2...\}$, and R^{δ} denote the Ruscheweyh derivative operator and given by

$$R^{\delta} = z + \sum_{k=2}^{\infty} c(\delta,k) a_k z^k \quad for \quad k, \delta \in \mathbb{N}_0, (z \in \mathbb{U}),$$

where $c(\delta,k) = \frac{(\delta+1)_{k-1}}{(1)_{k-1}}$.

If f is given by (1.1), then we can easily find from the equality (1.2) that

$$D^{\alpha,\delta}(m,q,\lambda) = z + \sum_{k=2}^{\infty} k^{\alpha}(1 + \frac{k-1}{1+q}\lambda)^{m} c(\delta,k) a_k z^k, \tag{1.3}$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 \qquad for \quad k = 0, x \in \mathbb{C} - \{0\}, \\ x(x+1)(x+2)...(x+k-1) \quad for \quad k \in \mathbb{N} = 1, 2, 3, ...and \quad x \in \mathbb{C}. \end{cases}$$

Special cases of this operator listed as the following:

- $D^{0,n}(0,q,\lambda) \equiv R^n$ [13] is the Ruscheweyh derivative operator.
- $D^{0,n}(\mu,0,\lambda) \equiv$ for ,$(m \in \mathbb{N}_0 = 0, 1, 2, ...)$, R^n_λ [8] is the generalized Ruscheweyh derivative operator.
- $D^{\alpha,0}(0,q,\lambda) \equiv D^{0,0}_1(m,0,1) \equiv S^n$ [4] is the Salagean derivative operator.

• $D^{0,0}(m,0,\lambda) \equiv S^n_\lambda$ [3] is the generalized Salagean derivative operator introduced by Al-Oboudi.

• $D^{0,\delta}(m,0,\lambda) \equiv D^n_\lambda$ [9] is the generalized Al-Shaqsi and Darus derivative operator.

• $D^{0,0}_1(n,\lambda,1) \equiv I_1(n,\lambda),(n \in \mathbb{Z})$, [5] is the operator investigated by Cho and Srivastava and also Cho and Kim [12].

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0) = 1, \quad \phi'(0) > 0$ which maps the open unit disc onto a region starlike with respect to 1 and is symmetric with respect to the real axis.

Let $S^*(\phi)$ be the class of functions $f \in S$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \qquad (z \in \mathbb{U}),$$

and let $C(\phi)$ be the class of functions $f \in S$ for which

$$1 + \frac{zf''(z)}{f(z)} \prec \phi(z), \qquad (z \in \mathbb{U}).$$

These classes were introduced and studied by Ma and Minda [15]. In the present paper, we obtain the Fekete-Szego inequality for functions in a more general class defined as follows:

Definition 1.2 *Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, $\phi(0) = 1, \quad \phi'(0) > 0$. A function $f \in A$ is in the class $S^{\alpha,\delta}(m,q,\lambda,\phi)$, if*

$$\frac{z(D^{\alpha,\delta}(m,q,\lambda)f(z))'}{D^{\alpha,\delta}(m,q,\lambda)f(z)} \prec \phi(z),$$

2 The Fekete-Szegö problem

In this section, we will give some upper bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$.

In order to prove our result, we have to recall the following lemmas:

Lemma 2.1 *[15] If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is an analytic function with positive real part in* $\mathbb{U}$*, then*

$$|c_2 - \upsilon c_1^2| \leq \begin{cases} -4\upsilon + 2 & if \quad \upsilon \leq 0, \\ 2 & if \quad 0 \leq \upsilon \leq 1, \\ 4\upsilon + 2 & if \quad \upsilon \geq 1. \end{cases}$$

When $\upsilon < 0$ or $\upsilon > 1$, the equality holds if and only if $p_1(z)$ is $\frac{(1+z)}{(1-z)}$, or one of its rotations. If $0 < \upsilon < 1$, then the equality holds if and only if $p_1(z)$ is $\frac{(1+z^2)}{(1-z^2)}$, or one of its rotations. If $\upsilon = 0$, the equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \leq a < 1),$$

or one of its rotations. If $\upsilon = 1$, the equality holds if and only if $p_1(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\upsilon = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \upsilon < 1$:

$$|c_2 - \upsilon c_1^2| + \upsilon |c_1| \leq 2, \qquad (0 < \upsilon \leq \frac{1}{2}),$$

and

$$|c_2 - \upsilon c_1^2| + (1 - \upsilon)|c_1| \leq 2, \qquad (\frac{1}{2} < \upsilon \leq 1).$$

Lemma 2.2 *[15] If* $p_1(z) = 1 + c_1 z + c_2 z^2 + \ldots$ *is an analytic function with positive real part in* $\mathbb{U}$*, then*

$$|c_2 - \upsilon c_1^2| \leq 2 \max\{1, |2\upsilon - 1|\}.$$

The result is sharp for the function

$$p_1(z) = \frac{(1+z)}{(1-z)} \quad or \quad p_1(z) = \frac{(1+z^2)}{(1-z^2)},$$

Theorem 2.1 *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *If* f *be given by (1.1) and belongs to the class* $S^{\alpha,\delta}(m,q,\lambda,\phi)$*. Then,*

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{B_1^2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{B_2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{\mu B_1^2(1+q)^m}{2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m} & if \quad \mu \leq \sigma_1, \\ \frac{B_1(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} & if \sigma_1 \leq \mu \geq \sigma_2, \\ -\frac{B_1^2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{B_2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{\mu B_1^2(1+q)^m}{2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m} & if \quad \mu \leq \sigma_2. \end{cases}$$

Where

$$\sigma_1 = \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)},$$

$$\sigma_2 = \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)}.$$

The result is sharp

Proof: For $f \in S^{\alpha,\delta}(m,q,\lambda,\phi)$, let

$$p(z) = \frac{z(D^{\alpha,\delta}(m,q,\lambda)f(z))'}{D^{\alpha,\delta}(m,q,\lambda)f(z)} = 1 + b_1 z + b_2 z^2 + \cdots . \tag{2.1}$$

From (2.1), we obtain

$$2^\alpha(\delta+1)\frac{(1+q+\lambda)^m}{(1+q)^m}a_2 = b_1,$$

and

$$3^\alpha(\delta+1)(\delta+2)\frac{(1+q+2\lambda)^m}{(1+q)^m}a_3 = b_2 + b_1 2^\alpha(\delta+1)\frac{(1+q+\lambda)^m}{(1+q)^m}a_2.$$

Since $\phi(z)$ is univalent and $p \prec \phi(z)$, the function

$$p_1(z) = \frac{1+\phi^{-1}(z)}{1-\phi^{-1}(z)} = 1 + c_1 z + c_2 z^2 + \cdots ,$$

is analytic and has positive real part in $\mathbb{U}$. Thus we have

$$p(z) = \phi(\frac{p_1(z)-1}{p_1(z)+1}), \tag{2.2}$$

and from this equation (2.2), we obtain

$$b_1 = \frac{1}{2}B_1 c_1 z,$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2)z^2.$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{\left[(B_1^2 c_1^2 + 2B_1[c_2 - \frac{1}{2}c_1^2] + B_2 c_1^2)(1+q)^m\right]}{4(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}$$
$$-\frac{\mu B_1^2 c_1^2 (1+q^{2m})}{2^{2(\alpha+1)}(\delta+1)^2(1+q+\lambda)^2 m},$$

$$a_3 - \mu a_2^2 = \frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}$$
$$\left[c_2 - c_1^2\left(\frac{1}{2}\left\{1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(1+q)^m(\delta+2)(1+q+2\lambda)^m}{2^{2\alpha}(\delta+1)(1+q+\lambda)^m}\right]\right)\right],$$

$$a_3 - \mu a_2^2 = \frac{B_1(1+q)^m}{2(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}[c_2 - \upsilon c_1^2],$$

where

$$\upsilon = \frac{1}{2}[1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(1+q)^\mu(\delta+2)(1+q+2\lambda)^m}{2^{2\alpha}(\delta+1)(1+q+\lambda)^m}].$$

If $\mu \leq \sigma_1$, then by applying Lemma 2.1, we get

$$|a_3 - \mu a_2^2| = \frac{B_1^2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{B_2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} -$$
$$\frac{\mu B_1^2(1+q)^m}{2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m},$$

if $\sigma_1 \leq \mu \leq \sigma_2$, we get

$$|a_3 - \mu a_2^2| = \frac{B_1(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m}.$$

Similarly, if $\mu \leq \sigma_2$, we get

$$|a_3-\mu a_2^2| = -\frac{B_1^2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{B_2(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{\mu B_1^2(1+q)^m}{2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m}.$$

If $\mu = \sigma_1$, then equality holds if and only if

$$p_1(z) = \left(\frac{1}{2}+\frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \leq a < 1, z \in \mathbb{U}).$$

or one of its rotations. Also, if $\mu = \sigma_2$, then

$$\frac{1}{2}[1 - B_1 - \frac{B_2}{B_1} + \mu\frac{3^\alpha(1+q)^\mu(\delta+2)(1+q+2\lambda)^m}{2^{2\alpha}(\delta+1)(1+q+\lambda)^m}] = 0.$$

Therefore,

$$\frac{1}{p_1(z)} = \left(\frac{1}{2}+\frac{1}{2}a\right)\frac{1+z}{1-z} + \left(\frac{1}{2}-\frac{1}{2}a\right)\frac{1-z}{1+z}, \quad (0 \leq a < 1, z \in \mathbb{U}).$$

Remark 2.1 *For* $q = 0$, $m = 1$ *and* $\alpha = 0$ *in Theorem* 2.1, *we get the results obtained by Al-Shaqsi and Darus [7]*

Remark 2.2 *If* $\sigma_1 \leq \mu \leq \sigma_2$, *then in view of Lemma* 2.1, *Theorem* 2.1 *can be improved. Let* σ_3 *be given by*

$$\sigma_3 = \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[B_2+B_1^2]}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)},$$

If the value $\sigma_1 \leq \mu \leq \sigma_3$, *then*

$$|a_3-\mu a_2^2| + \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)}$$

$$\left[B_1 - B_2 + \frac{(3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)+(\delta+1)(1+q+\lambda)^{2m}B_1^2)}{2^\alpha(\delta+1)(1+q+\lambda)^{2m}}\right]|a_2|^2 \leq$$

$$\frac{B_1(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m}.$$

If the value $\sigma_2 \le \mu \le \sigma_3$, *then*

$$|a_3-\mu a_2^2| + \frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)}$$

$$\left[B_1+B_2-\frac{(3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)+(\delta+1)(1+q+\lambda)^{2m}B_1^2)}{2^\alpha(\delta+1)(1+q+\lambda)^{2m}}\right]|a_2|^2 \le$$

$$\frac{B_1(1+q)^m}{3^\alpha(\delta+1)(\delta+2)(1+q+2\lambda)^m}.$$

Proof. For the values of $\sigma_1 \le \mu \le \sigma_3$, we have

$$|a_3-\mu a_2^2| + (\mu-\sigma_1)|a_2|^2$$

$$= \frac{B_1(1+q)^m}{2(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}[c_2-\upsilon c_1^2] +$$

$$(\mu-\sigma_1)\frac{B_1^2(1+q)^{2m}}{2^{2(\alpha+1)}(\delta+1)^2(1+q+\lambda)^2 m}|c_1|^2$$

$$= \frac{B_1(1+q)^m}{2(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}[c_2-\upsilon c_1^2] +$$

$$(\mu-\frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[B_1-B_2+B_1^2]}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)})\frac{B_1^2(1+q)^{2m}}{2^{2(\alpha+1)}(\delta+1)^2(1+q+\lambda)^2 m}|c_1|^2$$

$$= \frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left\{\frac{1}{2}[|c_2-\upsilon c_1^2| + \upsilon|c_1|^2]\right\}$$

$$\le \frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}.$$

Similarly, if $\sigma_2 \le \mu \le \sigma_3$, we can write

$$|a_3-\mu a_2^2| + (\sigma_2-\mu)|a_2|^2$$

$$= \frac{B_1(1+q)^m}{2(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}[c_2-\upsilon c_1^2] +$$

$$(\sigma_2-\mu)\frac{B_1^2(1+q)^{2m}}{2^{2(\alpha+1)}(\delta+1)^2(1+q+\lambda)^2m}|c_1|^2$$

$$=\frac{B_1(1+q)^m}{2(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}[c_2-\upsilon c_1^2]+$$

$$(\frac{2^\alpha(\delta+1)(1+q+\lambda)^{2m}[B_1+B_2+B_1^2]}{3^\alpha B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)}-\mu)\frac{B_1^2(1+q)^{2m}}{2^{2(\alpha+1)}(\delta+1)^2(1+q+\lambda)^2m}|c_1|^2$$

$$=\frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}\left\{\frac{1}{2}[|c_2-\upsilon c_1^2|+(1-\upsilon)|c_1|^2]\right\}$$

$$\leq\frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}.$$

Theorem 2.2 *Let $\phi(z)=1+B_1z+B_2z^2+\cdots$. If f be given by (1.1) and belongs to the class $S^{\alpha,\delta}(m,q,\lambda,\phi)$. For a complex number μ we have:,*

$$|a_3-\mu a_2^2|\leq\frac{B_1(1+q)^m}{(3^\alpha)(\delta+1)(\delta+2)(1+q+2\lambda)^m}$$

$$\max\{1,[-B_1-\frac{B_2}{B_1}+\mu\frac{3^\alpha(1+q)^\mu(\delta+2)(1+q+2\lambda)^m}{2^{2\alpha}(\delta+1)(1+q+\lambda)^m}]\}.$$

3 Applications of fractional derivatives

For fixed $g\in A$, let $S^{\alpha,\delta,g}(m,q,\lambda,\phi)$, be the class of functions $f\in A$ for which $(f*g)\in S^{\alpha,\delta}(m,q,\lambda,\phi)$. In order to introduce the class $S^{\alpha,\delta,\gamma}(m,q,\lambda,\phi)$, we need the following:

Definition 3.1 *[6] Let $f(z)$ be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order γ is defined by*

$$D_z^\gamma=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^z\frac{f(\zeta)}{(z-\zeta)^\gamma}d\zeta,\quad(0\leq\gamma<1),$$

where the multiplicity of $(z-\zeta)^\gamma$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta>0$ Using the above Definition 3.1 *and its known extensions* *involving fractional derivatives and fractional integrals, Owa and Srivastava [14] introduced the operator* $\Omega^{\gamma} : A \rightarrow A$ *defined by*

$$\Omega^{\gamma} f(z) = \Gamma(2-\gamma) z^{\gamma} D_z^{\gamma} f(z) \quad (\gamma \neq 2, 3, 4 \cdots).$$

The class $S^{\alpha,\delta,\gamma}(m, q, \lambda, \phi)$, *consists of functions* $f \in A$ *for which* $\Omega^{\gamma} \in S^{\alpha,\delta}(m, q, \lambda, \phi)$. *Note that* $S^{\alpha,\delta,\gamma}(m, q, \lambda, \phi)$, *is the special case of the class* $S^{\alpha,\delta,g}(m, q, \lambda, \phi)$. *when*

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} z^k,$$

$$g(z) = z + \sum_{k=2}^{\infty} g_k z^k.$$

Since $D^{\alpha,\delta}(m, q, \lambda) \in S^{\alpha,\delta,g}(m, q, \lambda)$ if and only if $D^{\alpha,\delta}(m, q, \lambda f(z)) * g(z) \in S^{\alpha,\delta}(m, q, \lambda, \phi)$. We obtain the coefficient estimate for functions in the class $S^{\alpha,\delta,g}(m, q, \lambda, \phi)$, from the corresponding estimate for functions in the class $S^{\alpha,\delta,g}(m, q, \lambda)$. Applying Theorem 2.1 for the function

$$D^{\alpha,\delta}(m, c, \lambda) * g(z) = z + \sum_{k=2}^{\infty} k^{\alpha} (1 + \frac{k-1}{1+q}\lambda)^m c(\delta, k) g_k a_k z^k.$$

we get the following Theorem 3.1 after an obvious change of the parameter μ

Theorem 3.1 *Let* $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$ *and* $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. *If* $D^{\alpha,\delta}(m, q, \lambda, \phi)$ *be given by (1.3) and belongs to the class* $S^{\alpha,\delta,g}(m, q, \lambda, \phi)$, *then*

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{1}{g_3}\left[\frac{B_1^2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{B_2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{\mu g_3 B_1^2(1+q)^m}{g_2^2 2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m}\right] & if \quad \mu \leq \sigma_1, \\ \frac{1}{g_3}\left[\frac{B_1(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\right] & if \sigma_1 \leq \mu \geq \sigma_2, \\ \frac{1}{g_3}\left[-\frac{B_1^2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{B_2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{g_3 \mu B_1^2(1+q)^m}{g_2^2 2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m}\right] & if \quad \mu \leq \sigma_2. \end{cases}$$

Where

$$\sigma_1 = \frac{g_2^2 2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{g_3 3^{\alpha} B_1^2 (1+q)^m (1+q+2\lambda)^m (\delta+2)},$$

$$\sigma_2 = \frac{g_2^2 2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{g_3 3^{\alpha} B_1^2 (1+q)^m (1+q+2\lambda)^m (\delta+2)}.$$

The result is sharp.

Since

$$\Omega^{\gamma} D^{\alpha,\delta}(m,q,\lambda) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\gamma)}{\Gamma(k+1-\gamma)} [k^{\alpha}(1 + \frac{k-1}{1+q}\lambda)^{m} c(\delta,k)] a_k z^k.$$

We have

$$g_2 = \frac{\Gamma(3)\Gamma(2-\gamma)}{\Gamma(3-\gamma)} = \frac{2}{(2-\gamma)},$$

$$g_3 = \frac{\Gamma(4)\Gamma(2-\gamma)}{\Gamma(4-\gamma)} = \frac{6}{(2-\gamma)(3-\gamma)},$$

For g_2 and g_3 given by the above equalities, Theorem 3.1 reduces to the following.

Theorem 3.2 *Let* $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, $(g_k > 0)$ *and* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$. *If* $D^{\alpha,\delta}(m,q,\lambda,\phi)$ *be given by (1.3) and belongs to the class* $S^{\alpha,\delta,g}(m,q,\lambda,\phi)$, *then* $|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{(2-\gamma)(3-\gamma)}{6}\left[\frac{B_1^2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{B_2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{\mu 3(2-\gamma)B_1^2(1+q)^m}{2(3-\gamma)2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m}\right] if \quad \mu \leq \sigma_1, \\ \frac{2-\gamma)(3-\gamma)}{6}\left[\frac{B_1(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m}\right] \qquad if \sigma_1 \leq \mu \geq \sigma_2, \\ \frac{2-\gamma)(3-\gamma)}{6}\left[-\frac{B_1^2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} - \frac{B_2(1+q)^m}{3^{\alpha}(\delta+1)(\delta+2)(1+q+2\lambda)^m} + \frac{3(2-\gamma)\mu B_1^2(1+q)^m}{2(3-\gamma)2^{2\alpha}(\delta+1)^2(1+q+\lambda)^m}\right] if \quad \mu \leq \sigma_2. \end{cases}$$

Where

$$\sigma_1 = \frac{(3-\gamma)2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}[B_1 - B_2 + B_1^2]}{3(2-\gamma)3^{\alpha}B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)},$$

$$\sigma_2 = \frac{(3-\gamma)2^{\alpha}(\delta+1)(1+q+\lambda)^{2m}[B_1 + B_2 + B_1^2]}{3(2-\gamma)3^{\alpha}B_1^2(1+q)^m(1+q+2\lambda)^m(\delta+2)}.$$

The result is sharp.

Remark 3.1 *When* $\lambda = 0$, $\delta = 0$, $m = 0$, $B_1 = \frac{8}{\pi^2}$, *and* $B_2 = \frac{16}{3\pi^2}$ *the above Theorem 3.2 reduces to a recent result of Srivastava and Mishra ([5], Theorem 8, p.64) for a class of functions for which* $\Omega^{\gamma} f(z)$ *is a parabolic starlike function [1].*

Note that other work related to generalized differential operators can be found in ([2] and [10]).

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