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SOME PROPERTIES OF A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY A GENERALIZED SRIVASTAVA-ATTIYA OPERATOR *

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Abstract. Making use of an integral operator which is defined by means of a general Hurwitz-Lerch Zeta function. This operator is a generalized Srivastava and Attiya operator. We give some properties of the class $Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$. Indeed, we obtain integral means inequalities, modified Hadamard products and establish some results concerning the partial sums for functions f belonging to the class $Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$.

Keywords: Analytic functions, Hurwitz-Lerch Zeta function, Srivastava-Attiya operator,Integral means, Hadamard product, Partial sums.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}). \tag{1.1}$$

We follow the similar approach Srivastava and Attiya operator using Hurwitz-Lerch Zeta function and Owa and Srivastava operator, the authors [1] have recently introduced a new generalized integral operator $\mathrm{Im}^{\alpha}_{s,b} f(z)$ as we will show in the following:

Definition 1.1. *(Srivastava and Choi [6]) A general HurwitzLerch Zeta function* $\Phi(z, s, b)$ *defined by*

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

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where $(s \in \mathbb{C},\, b \in \mathbb{C}-\mathbb{Z}_0^-)$ *when* $(|z|<1)$, *and* $(\mathrm{Re}(b)>1)$ *when* $(|z|=1)$.

Note that:

$$\Phi^*(z,s,b) = (b^s z\Phi(z,s,b)) * f(z) = z + \sum_{k=2}^{\infty} \frac{b^s}{(k+b-1)^s} a_k z^k.$$

Owa and Srivastava [2] introduced the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral as follows:

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k \qquad (\alpha \neq 2,3,4,\cdots),$$

where $D_z^\alpha f(z)$ the fractional derivative of f of order α (see [3]).

For $s \in \mathbb{C}$, $b \in \mathbb{C}-\mathbb{Z}_0^-$, and $0 \le \alpha < 1$, the generalized integral operator $(\mathrm{Im}_{s,b}^\alpha f) : \mathcal{A} \to \mathcal{A}$ is defined by

$$\begin{aligned} \mathrm{Im}_{s,b}^\alpha f(z) &= \Gamma(2-\alpha) z^\alpha D_z^\alpha \Phi^*(z,s,b), \quad (\alpha \neq 2,3,4,\cdots) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \left(\frac{b}{k-1+b}\right)^s a_k z^k, \quad (z \in \mathbb{U}). \end{aligned}$$

Note that : $\mathrm{Im}_{0,b}^0 f(z) = f(z)$.

Special cases of this operator includes:

- $\mathrm{Im}_{0,b}^\alpha f(z) \equiv \Omega^\alpha f(z)$ is the Owa and Srivastava operator [2].
- $\mathrm{Im}_{s,b+1}^0 f(z) \equiv J_{s,b} f(z)$ is the Srivastava and Attiya integral operator [4].
- $\mathrm{Im}_{\sigma,2}^0 f(z) \equiv I^\sigma f(z)$ is the Jung Kim Srivastava integral operator [5].

Also, the authors [1] have recently introduced a new subclass of analytic functions with negative coefficients, and stated as follows:

For $(0 \le \delta < 1)$, $(0 < \beta \le 1)$ and $(\frac{1}{2} < \gamma \le 1)$ if $\delta = 0$, and $(\frac{1}{2} < \gamma \le \frac{1}{2\delta})$if $\delta \neq 0$, we let $Q_{s,b}^\alpha(\delta,\beta,\gamma)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) and satisfying the inequality

$$\left| \frac{(\mathrm{Im}_{s,b}^\alpha f(z))' - 1}{2\gamma((\mathrm{Im}_{s,b}^\alpha f(z))' - \delta) - ((\mathrm{Im}_{s,b}^\alpha f(z))' - 1)} \right| < \beta.$$

We further let

$$Q^{*\alpha}_{s,b}(\delta,\beta,\gamma) = Q^{\alpha}_{s,b}(\delta,\beta,\gamma) \cap T, \tag{1.2}$$

where

$$T := \left\{ f \in \mathcal{A} : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \;\; where \;\; a_k \geq 0 \;\; for \; all \;\; k \geq 2 \right\},$$

is a subclass of $\mathcal{A}$ introduced and studied by Silverman [9].

In [1], it was also shown that the sufficient condition for a function f to be in the class $Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$.

Theorem 1.2. *Let the function f be defined by (1.2). Then $f \in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$ if and only if*

$$\sum_{k=2}^{\infty} k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right||a_k| \leq 2\beta\gamma(1-\delta). \tag{1.3}$$

The result is sharp.

2. Integral Means Inequalities

In order to prove the results regarding integral means inequalities, we need the concept of subordination between analytic functions and also the following lemma.

Lemma 2.1. *[8]*

If f, g are analytic in $\mathbb{U}$, such that $f \prec g$, then

$$\int_0^{2\pi} |f(z)|^y d\theta \leq \int_0^{2\pi} |g(z)|^y d\theta, \quad (z = re^{i\theta}, 0 < r < 1, y > 0).$$

Theorem 2.2. *Let $f \in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$. Then for $z = re^{i\theta}$, $0 < r < 1$, we have*

$$\int_0^{2\pi} |f(re^{i\theta})|^y d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^y d\theta, \quad (0 < r < 1, y > 0),$$

where the function $f_2(z)$ defined by

$$f_2(z) = z - \frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^{s}\right|} z^2. \tag{2.1}$$

Proof: Let $f \in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$ and satisfying (1.3), and $f_2(z)$ be given by (2.1). We must show that

$$\int_0^{2\pi}\left|1-\sum_{k=2}^{\infty}a_k z^{k-1}\right|d\theta \le \int_0^{2\pi}\left|1-\frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|}z\right|^y d\theta.$$

By Lemma 2.1, it suffices to show that

$$1-\sum_{k=2}^{\infty}a_k z^{k-1} \prec 1-\frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|}z.$$

Setting

$$1-\sum_{k=2}^{\infty}a_k z^{k-1} = 1-\frac{2\beta\gamma(1-\delta)}{2\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|}\omega(z). \tag{2.2}$$

From (2.2), we obtain

$$|\omega(z)| \le |z|\sum_{k=2}^{\infty}\frac{2\beta\gamma(1-\delta)}{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}|a_k| \le |z| < 1.$$

This completes the proof of the theorem.

3. Modified Hadamard Products

Let the functions $f_j(z)(j=1;2)$ be defined by

$$f_j(z) = z-\sum_{k=2}^{\infty}a_{k,j}z^k, \quad for \quad all \quad (a_{k,j}\ge\ 0, z\in\mathbb{U}). \tag{3.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z-\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k.$$

Using the techniques of Schild and Silverman [7], we prove the following results.

Theorem 3.1. *For functions $f_j(z)(j=1;2)$ defined by (3.1), let $f_1(z)\in Q^{*\alpha}_{s,b}(\delta,\beta,\gamma)$, $f_2(z)\in Q^{*\alpha}_{s,b}(\delta,\mu,\gamma)$. Then $(f_1*f_2)(z)\in \xi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta,\beta,\mu,\gamma))$, where*

$$\xi^{*\alpha}_{s,b}(\delta,Q^{*\alpha}_{s,b}(\delta,\beta,\gamma) = 1-\frac{2\gamma(1-\delta)\beta\mu}{4\beta\mu\gamma^2(1-\delta)-2\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))}.$$

Proof: To prove the theorem, we need to find the largest $\xi = \xi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta,\beta,\mu,\gamma))$ such that

$$\sum_{k=2}^{\infty} \frac{k\left[1+\xi(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\xi\gamma(1-\delta)} a_{k,1}a_{k,2} \leq 1,$$

since

$$\sum_{k=2}^{\infty} \frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\beta\gamma(1-\delta)} a_{k,1} \leq 1,$$

and

$$\sum_{k=2}^{\infty} \frac{k\left[1+\mu(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\mu\gamma(1-\delta)} a_{k,2} \leq 1.$$

By the Cauchy-Schwarz inequality, we have;

$$\sum_{k=2}^{\infty} \frac{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|}{2\gamma(1-\delta)} \sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu} a_{k,1}a_{k,2}} \leq 1.$$

Thus, it is suffices to show that

$$\frac{(1+\xi(2\gamma-1))}{\xi} a_{k,1}a_{k,2} \leq \sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu} a_{k,1}a_{k,2}}.$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{2\gamma(1-\delta)}{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|} \sqrt{\frac{\beta}{(1+\beta(2\gamma-1))}\frac{\mu}{(1+\mu(2\gamma-1))}}.$$

Consequently, we need only to prove that

$$\frac{2\gamma(1-\delta)}{k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|} \sqrt{\frac{\beta}{(1+\beta(2\gamma-1))}\frac{\mu}{(1+\mu(2\gamma-1))}} \leq$$

$$\frac{\xi}{(1+\xi(2\gamma-1))} \sqrt{\frac{(1+\beta(2\gamma-1))}{\beta}\frac{(1+\mu(2\gamma-1))}{\mu}},$$

or, equivalently that

$$\xi \leq 1 - \frac{2\gamma(1-\delta)\beta\mu}{4\beta\mu\gamma^{2}(1-\delta) - k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^{s}\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))} = \psi(k),$$

is an increasing function of k, letting $k = 2$, we obtain

$$\psi(2) = 1 - \frac{2\gamma(1-\delta)\beta\mu}{4\beta\mu\gamma^2(1-\delta) - 2\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|(1+\mu(2\gamma-1))(1+\beta(2\gamma-1))},$$

which completes the proof.

Theorem 3.2. *For functions $f_j(z)(j = 1; 2)$ defined by (3.1), be in the class $Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$ Then the function $h(z) = z - \sum_{k=2}^{\infty}\left(a^2_{k,1} + a^2_{k,2}\right)z^k$, belongs to the class $\varphi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$, where*

$$\varphi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta, \beta, \gamma) = 1 - \frac{4\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|(2\beta\gamma(1-\delta))^2}{2\gamma(1-\delta)[2(1+\beta(2\gamma-1))\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|]}.$$

Proof: By virtue of Theorem 1.2, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}\right]^2 a^2_{k,1} \le 1, \tag{3.2}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}\right]^2 a^2_{k,2} \le 1. \tag{3.3}$$

It follows from (3.2) and(3.3).

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}\right]^2 (a^2_{k,1} + a^2_{k,2}) \le 1.$$

Therefore,we need to find the largest $\varphi = \varphi^{*\alpha}_{s,b}(\delta, Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$

$$\frac{k\left[1+\varphi(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\varphi\gamma(1-\delta)} \le \frac{1}{2}\left[\frac{k\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}\right]^2,$$

that is,

$$\varphi \le 1 - \frac{2k\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|(2\beta\gamma(1-\delta))^2}{2\gamma(1-\delta)[k(1+\beta(2\gamma-1))\left(\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}\right)\left|\left(\frac{b}{k-1+b}\right)^s\right|]} = \chi(k),$$

is an increasing function of k, letting $k = 2$, we obtain

$$\chi(2) = 1 - \frac{4\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|(2\beta\gamma(1-\delta))^2}{2\gamma(1-\delta)[2(1+\beta(2\gamma-1))\left(\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\right)\left|\left(\frac{b}{1+b}\right)^s\right|]},$$

which completes the proof.

4. Partial sums

By following the earlier works by Silverman[10] on partial sums of analytic functions, we study the ratio of a function of the form (1.2) to its sequence of partial sums of the form $f_1(z) = z$, $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$, $\quad (z \in \mathbb{U})$.

We will determine sharp lower bounds for

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\}, \operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}, \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \quad \textit{and} \quad \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\}.$$

Theorem 4.1. *Let $f \in Q_{s,b}^{*\alpha}(\delta, \beta, \gamma)$ and satisfying (1.3), then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq 1 - \frac{1}{c_{n+1}}, \quad (n \in \mathbb{N}, \quad z \in \mathbb{U}), \tag{4.1}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{1 + c_{n+1}}, \quad (n \in \mathbb{N}, \quad z \in \mathbb{U}), \tag{4.2}$$

where c_n be defined as

$$c_n = \frac{n\left[1 + \beta(2\gamma - 1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}.$$

The results are sharp for every k with the function given by

$$f(z) = z - \frac{z^{n+1}}{c_{n+1}}, \quad (z \in \mathbb{U}, n \in \mathbb{N}). \tag{4.3}$$

Proof: In order to prove (4.1), it is sufficies to show that

$$c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\} = \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} = \frac{1 + w(z)}{1 - w(z)}.$$

Then

$$w(z) = \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}.$$

Notice that $w(0) = 0$ and

$$|w(z)| = \frac{c_{n+1}\sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{n} |a_k| |z^{k-1} - \sum_{k=n+1}^{\infty} c_{n+1}|a_k|}.$$

Now $|w(z)| \leq 1$ if and only if

$$c_{n+1}\sum_{k=n+1}^{\infty} |a_k| + \sum_{k=2}^{n} |a_k| \leq 1. \tag{4.4}$$

It suffices to show that the LHS of (4.4) is bounded above by the condition (1.3) which is equivalent to

$$\sum_{k=n+1}^{\infty} (c_k - c_{n+1})\,|a_k| + \sum_{k=2}^{n} (c_k - 1)\,|a_k| \geq 0.$$

To see that the function given by (4.3) gives the sharp result, we observe that for $z = re^{\frac{\pi i}{n}}$,

$$\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \to 1 - \frac{1}{c_{n+1}}, \qquad when \quad (z \to 1^-).$$

To prove the second part of this theorem, we write

(4.5)

$$(1 + c_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{c_{n+1} + 1}\right\} = \frac{1 + \sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} c_{n+1} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} = \frac{1 + w(z)}{1 - w(z)},$$

we find that

$$w(z) = \frac{\sum_{k=n+1}^{\infty} (1 + c_{n+1})\, a_k z^{k-1}}{2 + 2\sum_{k=2}^{n} a_k z^{k-1} + \sum_{k=n+1}^{\infty} (1 + c_{n+1})\,|a_k z^{k-1}}.$$

Now $|w(z)| \leq 1$ if and only if

$$(1 + c_{n+1})\sum_{k=n+1}^{\infty} |a_k| + \sum_{k=2}^{n} |a_k| \leq 1.$$

The equality holds in (4.2) for the extremal function f given by (4.3).

This completes the proof.

Theorem 4.2. *Let* $f \in Q^{*\alpha}_{s,b}(\delta, \beta, \gamma)$ *and satisfying (1.3), then*

$$\mathrm{Re}\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq 1 - \frac{n+1}{c_{n+1}}, \quad (z \in \mathbb{U}), \tag{4.6}$$

and

$$\mathrm{Re}\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{n+1+c_{n+1}}. \tag{4.7}$$

The results are sharp with the function given by (4.3).

Proof: To prove the result (4.6), define the function $w(z)$ by

$$c_{n+1}\left\{\frac{f'(z)}{f'_n(z)} - (1 - \frac{n+1}{c_{n+1}})\right\} = \frac{1+w(z)}{1-w(z)}.$$

Then

$$w(z) = \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty} k a_k}{2 + 2\sum_{k=2}^{n} k a_k z^{k-1} + \sum_{k=n+1}^{\infty} \frac{c_{n+1}}{n+1} k}.$$

Now $|w(z)| \leq 1$ if and only if

$$\left(\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k|a_k| + \sum_{k=2}^{n} k|a_k| \leq 1.$$

From the condition (1.3), it is sufficies to show that

$$\left(\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty} k|a_k| + \sum_{k=2}^{n} k|a_k| \leq c_k|a_k|.$$

This is equivalent to showing that

$$\sum_{k=2}^{n}(c_k - k)|a_k| + \sum_{k=n+1}^{\infty} \frac{(n+1)c_k - kc_{n+1}}{n+1}|a_k| \geq 0.$$

To prove the second part of this theorem, we write

$$w(z) = (n+1+c_{n+1})\left\{\frac{f'_n(z)}{f'(z)} - (\frac{c_{n+1}}{n+1+c_{n+1}})\right\} = 1 - \frac{(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1+\sum_{k=2}^{n} k a_k z^{k-1}},$$

yields

$$\left|\frac{w(z)-1}{w(z)+1}\right| \leq \frac{(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k|}{2-2\sum_{k=2}^{n} k|a_k| - (1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k|} \leq 1, \quad (z \in \mathbb{U}),$$

if and only if

$$2(1+\frac{c_{n+1}}{n+1})\sum_{k=n+1}^{\infty} k|a_k| \leq 2-2\sum_{k=2}^{n} k|a_k|.$$

The bound in (4.7) is sharp for all $n \in \mathbb{N}$ with the extremal function (4.3).

This completes the proof of theorem.

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