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Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients

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خصائص المعامل التفاضلي المعمم على فئة فرعية معينة من الدوال التحليلية ذات معاملات سالبة

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الملخص:

الهدف الرئيسي من هذه الورقة البحثيه هو تقديم ودراسة فئة فرعية جديدة من الدوال التحليلية المحددة من قبل المعامل التفاضلي المعمم مع معاملات السالبة في قرص الوحدة . وتشمل النتائج حساب المعاملات ، ونقاط المتطرفة ، النمو و التشويه ومعامل تكاملي في مجال الفئات الفرعية المذكور.

Properties of Generalized Derivative Operator to A Certain Subclass of Analytic Functions with Negative Coefficients

Abstract.

The main object of this paper is to introduce and study the new subclasses $\mathrm{T}^{\alpha,n}(m,q,\lambda,\beta)$ and $\mathrm{C}^{\alpha,n}(m,q,\lambda,\beta)$ of analytic functions defined by generalized derivative operator with negative coefficients in the unit disk. The results presented here include coefficient estimates, extreme points, growth and distortion properties and integral operators for the aforementioned subclasses.

Keywords: starlike; convex; distortion theorems; derivative operator.

1. Introduction.

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathrm{U} = \{z : |z| < 1\}$. A function $f \in \mathrm{A}$ is called starlike functions of order β and type γ, if it satisfies

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\gamma} \right| < \beta, \tag{2}$$

where $0 \le \gamma < 1,\ 0 < \beta \le 1$. We denote by $\mathrm{S}^*(\gamma, \beta)$ the class of starlike functions of order γ and type β. A function $f \in \mathrm{A}$ is called convex functions of order β and type γ, if it satisfies

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2 - 2\gamma} \right| < \beta, \tag{3}$$

where $0 \le \gamma < 1,\ 0 < \beta \le 1$. We denote by $\mathrm{K}(\gamma, \beta)$ the class of convex functions of order γ and type β.

From (2) and (3), we note that: $f(z) \in \mathrm{K}(\gamma, \beta)$ if, and only if,

$$zf' \in \mathrm{S}^*(\gamma, \beta).$$

The classes $\mathrm{S}^*(\gamma, \beta)$ and $\mathrm{K}(\gamma, \beta)$ were considered by Gupta and Jain [9].

Let T denote the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad (a_k \ge 0,). \tag{4}$$

We denote by $\mathrm{T}^*(\gamma, \beta)$ and $\mathrm{C}(\gamma, \beta)$, the classes obtained by taking intersections, respectively, of the classes $\mathrm{S}^*(\gamma, \beta)$ and $\mathrm{K}(\gamma, \beta)$ with the class T. Thus we have

$$T^*(\gamma,\beta) = S^*(\gamma,\beta) \cap T,$$

and

$$C(\gamma,\beta) = K(\gamma,\beta) \cap T.$$

The classes $T^*(\gamma,\beta)$ and $C(\gamma,\beta)$ were studied by [10] Also the classes $T^*(\gamma,1) = T^*(\gamma)$ and $C(\gamma,1) = C(\gamma)$ were studied by Silverman [8].

For functions $f \in A$, given by (1), and g given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product(or convolution)of functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$$

The theory of derivative play an important role in the theory of univalent functions. It is believed that Ruscheweyh (1975) was the first to give a generalised derivative operator in the theory of univalent function. Later, S‹al‹agean (1983) gave another generalised derivative operator. In the same paper, he introduced an integral operator. Many properties have been discussed and studied by many researchers for these two operators. For example, Al-Oboudi (2004) introduced a generalised Salagean operator, Al-Shaqsi and Darus (2009) generalised the operator given by Ruscheweyh (1975), while Darus and Al-Shaqsi (2008) studied both derivatives of Ruscheweyh and Salagean. These operators motivate us to create another type of derivative operator.

The author in [1] have recently introduced a new generalised derivative operator $D^{\alpha,n}(m,q,\lambda)f(z)$ as the following:

For the function $f \in A$ given by (1) we define a new generalised derivative operator $D^{\alpha,n}(m,q,\lambda)f(z): A \to A$ as follows:

$$D^{\alpha,n}(m,q,\lambda)(f)(z) = z + \sum_{k=2}^{\infty} k^{\alpha}\left(1+\frac{k-1}{1+q}\lambda\right)^m c(n,k) a_k z^k, \qquad (5)$$

where $n,\alpha \in \mathrm{N}_0 = \{0,1,2...\}, m \in \mathrm{Z}$, $\lambda, q \geq 0$ and $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $m = 0,1,2,...$, then

$$\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z) = \underbrace{\phi(z)*...*\phi(z)}_{(m)-times} * \left[\frac{z}{(1-z)^{n+1}}\right] * \sum_{k=1}^{\infty} k^{\alpha} z^{k} * f(z)$$

$$= R^{n} * \mathrm{D}^{\alpha}(m,q,\lambda)f(z),$$

where $R^{n} = z + \sum_{k=2}^{\infty} c(n,k) z^{k}$, the Ruscheweyh derivative operator.

If $m = -1,-2,...$, then

$$\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z) = \underbrace{\phi(z)*...*\phi(z)}_{(-m)-times} * \left[\frac{z}{(1-z)^{n+1}}\right] * \sum_{k=1}^{\infty} k^{\alpha} z^{k} * f(z)$$

$$= R^{n} * \mathrm{D}^{\alpha}(m,q,\lambda)f(z).$$

Note that:

$$\mathrm{D}^{0,0}(0,q,\lambda)f(z) = \mathrm{D}^{0,0}(1,0,0)f(z) = f(z), \quad and$$

$$\mathrm{D}^{0,0}(1,q,\lambda)f(z) = zf'(z).$$

By specialising the parameters of $\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z)$, we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [2];

$$\mathrm{D}^{0,n}(0,q,\lambda) \equiv \mathrm{D}^{0,n}(1,0,0); (n \in \mathrm{N}_0) \equiv R^{n} = z + \sum_{k=2}^{\infty} c(n,k) a_k z^{k}.$$

- The derivative operator introduced by Sălăgean [3];

$$\mathrm{D}^{\alpha,0}(0,q,\lambda) \equiv \mathrm{D}_1^{0,0}(n,0,1); (n \in \mathrm{N}_0) \equiv D^{n} = z + \sum_{k=2}^{\infty} k^{n} a_k z^{k}.$$

- The generalised Salagean derivative operator introduced by Oboudi [4];

$$D^{0,0}(n,0,\lambda);(n\in N_0)\equiv D_\lambda^n=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^n a_k z^k.$$

- The generalised Ruscheweyh derivative operator introduced by Darus and Al-Shaqsi [5];

$$D^{0,n}(1,0,\lambda);(n\in N_0)\equiv R_\lambda^n=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))c(n,k)a_k z^k.$$

- The derivative operator introduced by Catas [6];

$$D^{0,\beta}(m,l,\lambda);(m\in N_0)\equiv D^m(\lambda,\beta,l)=z+\sum_{k=2}^{\infty}(\frac{1+\lambda(k-1)+l}{1+l})^m c(\beta,k)a_k z^k.$$

- The integral operator introduced by Cho and T. H. Kim [7];

$$D^{1,0}(-n,\lambda,1)\equiv I_n^\lambda=z+\sum_{k=2}^{\infty}k(\frac{1+\lambda}{k+\lambda})^n a_k z^k.$$

Next we define the following new subclasses functions as follows:

Definition 0.1 *Let* $f\in T$ *be given by (4). Then* f *is said to be in the class* $T^{\alpha,n}(m,q,\lambda,\beta)$ *if, and only if,*

$$\left|\frac{\dfrac{z(D^{\alpha,n}(m,q,\lambda)f)'(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}-1}{\dfrac{z(D^{\alpha,n}(m,q,\lambda))f)'(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}+1-2\gamma}\right|<\beta,$$

where $D^{\alpha,n}(m,q,\lambda)f(z)$ is given by (5) and $n,\alpha\in N_0=\{0,1,2...\},m\in Z,$ $\lambda,q\geq 0$ and $0\leq\gamma<1,$ $0<\beta\leq 1.$ Further, a function $f\in T$ is said to be in the class $C^{\alpha,n}(m,q,\lambda,\beta)$ if, and only if,

$$zf'\in T^{\alpha,n}(m,q,\lambda,\beta).$$

We note that, by specializing the parameters α,n,m,λ,β we shall obtain the following subclasses which were studied by various authors:

1. For $\alpha=n=m=0,$ we have $T^{0,0}(0,q,\lambda,\beta)=S^*(\gamma,\beta),$ is the class of starlike function of order γ and type β which was studied by Gupta and Jain [9].

2 . For $\alpha = n = m = 0$ and $\beta = 1$ we obtain the class $T^{0,0}(0,q,\lambda,1) = T^*(\gamma)$, which was studied by Silverman [8].

3. For $\alpha = n = q = 0, m = 1$ we have the class $C^{0,0}(1,0,\lambda,\beta) = C(\gamma,\beta)$, which was studied by Gupta and Jain [9].

4. For $\alpha = n = q = 0, m = 1, \beta = 1,$ we obtain the class $C^{0,0}(1,0,\lambda,1) = C(\gamma)$, studied by Silverman [8].

Main Results.

In this paper we introduce, .coefficient Inequalities,growth and distortion Properties, extreme points and the class preserving integral operators of the form:

$$F(z) := \frac{(c+1)}{z^c}\int_0^z t^{c-1} f(t), \tag{6}$$

for the subclasses $T^{\alpha,n}(m,q,\lambda,\beta)$ and $C^{\alpha,n}(m,q,\lambda,\beta)$ are considered.

2. Coefficient Inequalities

In this segment, we give an important and adequate condition for a function $f(z)$, given by (1), to be in subclasses $T^{\alpha,n}(m,q,\lambda,\beta)$ and $C^{\alpha,n}(m,q,\lambda,\beta)$.

Theorem 2.1. *A function* f *belongs to the subclass* $T^{\alpha,n}(m,q,\lambda,\beta)$ *if, and only if,*

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k\right) \le 2\beta(1-\gamma). \tag{7}$$

Proof. Let the function f be in the class $T^{\alpha,n}(m,q,\lambda,\beta)$. Then we have

$$\left|\frac{\frac{z(\mathrm{D}^{\alpha,n}(m,q,\lambda)f)'(z)}{\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z)}-1}{\frac{z(\mathrm{D}^{\alpha,n}(m,q,\lambda)f)'(z)}{\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z)}+1-2\gamma}\right|=\left|\frac{\frac{z-\sum_{k=2}^{\infty}(k)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k z^k}{z-\sum_{k=2}^{\infty}(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k z^k}-1}{\frac{z-\sum_{k=2}^{\infty}(k)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k z^k}{z-\sum_{k=2}^{\infty}(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+\delta)}a_k z^k}+1-2\gamma}\right|\le\beta.$$

Since $|Re(z)|\le|z|$ for all z, we have

$$\Re\left\{\frac{\sum_{k=2}^{\infty}(k-1)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+\delta)}{k!\Gamma(1+n)}a_k z^k}{-\sum_{k=2}^{\infty}(k+1-2\gamma)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k z^k+(2-2\gamma)}\right\}\le\beta.$$

Choosing values of z on the real axis, so that $\frac{z(\mathrm{D}^{\alpha,n}(m,q,\lambda)f)'(z)}{\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z)}$ is real, and letting $z\to 1^{-}$, through real axis, we get

$$\sum_{k=2}^{\infty}(k-1)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{(k)!\Gamma(1+n)}a_k z^k\le$$

$$-\beta(\sum_{k=2}^{\infty}(k+1-2\gamma)(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{\mu}\frac{\Gamma(k+\delta)}{k!\Gamma(1+\delta)}a_k z^k+\beta(2-2\gamma)),$$

which implies the assertion (7). Conversely, let the inequality (7) holds true, then

$$\left|z(\mathrm{D}^{\alpha,n}(m,q,\lambda)f)'(z)-(\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z))\right|-\beta$$

$$\left|z(\mathrm{D}^{\alpha,n}(m,q,\lambda)f)'(z)+(1-2\gamma)\mathrm{D}^{\alpha,n}(m,q,\lambda)f(z)\right|,$$

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+\delta)}{k!\Gamma(1+\delta)}\right)-\beta(2-2\gamma)\le 0,$$

by the assumption. This implies that $f\in\mathrm{T}^{\alpha,n}(m,q,\lambda,\beta)$

Corollary 2.2. *Let the function* f *be in the class* $\mathrm{T}^{\alpha,n}(m,q,\lambda,\beta)$, *then*

$$a_k \le \frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}}. \tag{8}$$

The result (8) is sharp for the function f of the form

$$f(z) = z - \frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}} z^k. \tag{9}$$

By using the same arguments as in the proof of Theorem 2.1, we can establish the next theorem.

Theorem 2.3. *A function* f *belongs to the subclass* $C^{\alpha,n}(m,q,\lambda,\beta)$, *if, and only if,*

$$\sum_{k=2}^{\infty}\left(k[(k-1)+\beta(k+1-2\gamma)](k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)} a_k z^k \right) \le 2\beta(1-\gamma),$$

Corollary 2.4. *Let the function* f *be in the class* $C^{\alpha,n}(m,q,\lambda,\beta)$. *Then*

$$a_k \le \frac{2\beta(1-\gamma)}{k[(k-1)+\beta(k+1-2\gamma)](k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}},$$

with equality only for functions of the form

$$f(z) = z - \frac{2\beta(1-\gamma)}{k[(k-1)+\beta(k+1-2\gamma)](k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m \frac{\Gamma(k+n)}{k!\Gamma(1+n)}} z^k.$$

3. Growth and distortion theorems

In this segment, we obtain growth and distortion bounds for the classes $T^{\alpha,n}(m,q,\lambda,\beta)$ and $C^{\alpha,n}(m,q,\lambda,\beta)$.

Theorem 3.1. *If* $f \in T^{\alpha,n}(m,q,\lambda,\beta)$, *then*

$$|f(z)| \ge r - \frac{2\beta(1-\gamma)}{(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^m \frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}} r^2 \tag{10}$$

$$\leq r+\frac{2\beta(1-\gamma)}{(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}}r^{2}, \tag{11}$$

and

$$|f^{'}(z)|\geq 1-\frac{2\beta(1-\gamma)(2)}{(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(p+1)!\Gamma(1+n)}}r \tag{12}$$

$$\leq 1+\frac{2\beta(1-\gamma)(2)}{(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(p+1)!\Gamma(1+n)}}r, \tag{13}$$

for $z\in U$. The estimates for $|f(z)|$ and $|f^{'}(z)|$ are sharp.

Proof. Since $f\in T^{\alpha,n}(m,q,\lambda,\beta)$, and in view of inequality (7) of Theorem 2.1, we have

$$(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}\sum_{k=2}^{\infty}a_k\leq$$

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k z^{k}\right)\leq 2\beta(1-\gamma),$$

or

$$\sum_{k=2}^{\infty}a_k\leq\frac{2\beta(1-\gamma)}{(1+\beta(3-2\gamma))(2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}}. \tag{14}$$

Since

$$r-r^{2}\sum_{k=2}^{\infty}a_k\leq|f(z)|\leq r+r^{2}\sum_{k=2}^{\infty}a_k, \tag{15}$$

on using (14) and (15), we easily arrive at the desired results of (11) and (10). Furthermore, we observe that

$$1-(2)r\sum_{k=2}^{\infty}a_k\leq|f^{'}(z)|\leq 1+(2)r\sum_{k=2}^{\infty}a_k, \tag{16}$$

On using (14) and (16), we easily arrive at the desired results of (12) and (13). Finally, we can see that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp for the function,

$$f(z)=z^p-\frac{2\beta(p-\gamma)}{(1+(1+2p-2\gamma))(1+\frac{\lambda}{p+q})^{\mu}\frac{\Gamma(p+1+\delta)}{(p+1)!\Gamma(p+\delta)}}.$$

Similarly, we can prove the following theorem.

Theorem 3.2. *If* $f\in C^{\alpha,n}(m,q,\lambda,\beta)$, *then*

$$|f(z)|\geq r-\frac{2\beta(1-\gamma)}{(2)[1+\beta(3-2\gamma)](2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}}r^2$$

$$\leq r+\frac{2\beta(1-\gamma)}{(2)[1+\beta(2-2\gamma)](2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(1+n)}}r^2,$$

and

$$|f'(z)|\geq 1-\frac{4\beta(1-\gamma)}{[1+\beta(3-2\gamma)](2)^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(2+n)}{(2)!\Gamma(2+n)}}r$$

$$\leq 1+\frac{4\beta(1-\gamma)}{(2)[1+\beta(3-2\gamma)](2\frac{p+1}{p})^{\alpha}(1+\frac{\lambda}{1+q})^{m}\frac{\Gamma(3+n)}{(2)!\Gamma(1+n)}}r,$$

for $z\in U$. The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

4. Extreme Points Now, we determine extreme points for the subclasses $T^{\alpha,n}(m,q,\lambda,\beta)$ and $C^{\alpha,n}(m,q,\lambda,\beta)$.

Theorem 4.1. *Let* $f(z)=z$ *and,*

$$f_k(z)=z-\frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}z^k.$$

Then f is in the class $T_p^{\alpha,n}(m,q,\lambda,\beta)$, if ,and only if, it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\omega_k f_k(z),$$

where

$$\omega_k\geq 0,\sum_{k=0}^{\infty}\omega_k=1. \tag{17}$$

Proof. Let $f(z)=\sum_{k=0}^{\infty}\omega_k f_k(z)$

$$f(z)=z-\frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}\omega_k z^k.$$

Then, in view of (17), it follows that

$$\sum_{k=2}^{\infty}\frac{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}{2\beta(1-\gamma)}\times$$

$$\left\{\frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}\omega_k\right\}=\sum_{k=1}^{\infty}\omega_k=1-\omega_1\leq 1.$$

Thus $f\in T^{\alpha,n}(m,q,\lambda,\beta)$.

Conversely, assume that a function f defined by (4) belongs to class $T^{\alpha,n}(m,q,\lambda,\beta)$. Then

$$a_k\leq\frac{2\beta(1-\gamma)}{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}.$$

We set

$$\omega_k=\frac{((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{(k-1)}{p+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}}{2\beta(1-\gamma)},$$

and $\omega_k=1-\sum_{k=1}^{\infty}\omega_k$. Then we have $f(z)=\sum_{k=1}^{\infty}\omega_k f_k(z)$. This completes the proof.

Similarly, we can prove the following result:

Theorem 4.2. *Let* $f(z)=z$ *and,*

$$f_k(z)=z-\frac{2\beta(1-\gamma)}{k[(k-1)+\beta(k+1-2\gamma)](k)^{\alpha}(1+\frac{(k-1)}{1+q}\lambda)^m\frac{\Gamma(k+n)}{(k)!\Gamma(1+n)}}z^k.$$

Then f is in the class $C^{\alpha,n}(m,q,\lambda,\beta)$, if, and only if, it can be expressed in the form

$$f(z)=\sum_{k=0}^{\infty}\omega_k f_k(z),$$

where

$$\omega_k\geq 0,\sum_{k=0}^{\infty}\omega_k=1.$$

5. Integral Operators.

Theorem 5.1 If the function $f(z)$ given by (4) is in the subclass $T^{\alpha,n}(m,q,\lambda,\beta)$, where $0\leq\gamma<1$, $0<\beta\leq 1$, $n,\alpha\in N_0=\{0,1,2...\}, m\in Z$, $\lambda,q\geq 0$ and let c be a real number such that $c>-1$. f belongs to the class $T^{\alpha,n}(m,q,\lambda,\beta)$, then the function F defined by

$$F(z):=\frac{(c+1)}{z^c}\int_0^z t^{c-1}f(t), \qquad (18)$$

also belongs to $T^{\alpha,n}(m,q,\lambda,\beta)$.

Proof. Let $f\in T$ Then from representation of F, it follows that

$$F(z)=z-\sum_{k=2}^{\infty}b_k z^k, \qquad (b_k\geq 0).$$

where $b_k=\left(\frac{c+1}{c+k}\right)a_k$. Therefore using Theorem 2.1 for the coefficients of F, we have

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^m\frac{\Gamma(k+n)}{k!\Gamma(1+n)}b_k\right)=$$

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}\left(\frac{c+1}{c+k}\right)a_k\right)\leq$$

$$\sum_{k=2}^{\infty}\left(((k-1)+\beta(k+1-2\gamma))(k)^{\alpha}(1+\frac{k-1}{1+q}\lambda)^{m}\frac{\Gamma(k+n)}{k!\Gamma(1+n)}a_k\right)\leq 2\beta(1-\gamma).$$

since $\left(\frac{c+1}{c+k}\right)<1$ and $f\in \mathrm{T}^{\alpha,n}(m,q,\lambda,\beta)$. Hence $F\in \mathrm{T}^{\alpha,n}(m,q,\lambda,\beta)$.

Many other work on analytic functions functions related to derivative operator and integral operator can be read in [11], [12]

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