



Differential Subordination and superordination Results Associated with Generalized Derivative operator

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نتائج دوال التبعية التفاضلية والتي ترتبط مع المعامل التفاضلي المعمم

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الملخص:

باستخدام أساليب التبعية التفاضلية تم الحصول على نتائج دوال التبعية تفاضلية للدوال التحليلية في القرص وحدة مفتوحة والتي ترتبط مع المعامل التفاضلي المعمم. ويتم الحصول على هذه النتائج من خلال التحقيق من فئة فرعية لدوال المسلم بها. الحصول أيضا على النتائج نظرية شطيرة. بعض النتائج المقدمة في هذه الورقة مبنية على توسيع لبعض النتائج المعطى حديثا.

Abstract

Using the methods of differential subordination and superordination, sufficient conditions are determined on the generalized derivative operator for analytic functions in the open unit disc U to obtain, these results are obtained by investigating appropriate classes of admissible functions. New differential sandwich-type results are also obtained for the generalized derivative. Some of the results established in this paper would provide extensions of those given in earlier works.

Keywords: Analytic function, Differential subordination, Superordination.

1. Introduction, Definitions, and Preliminaries

The theory of differential subordination in the complex plane is the generalization of a differential inequality on the real line, which began with the remarkable article "Differential subordination and univalent functions" by Miller, Mocanu (1981). Since then, hundreds of papers have appeared in the literature on this topic. The applications and extensions of the theory have been developed in other fields like differential equations, partial differential equations, meromorphic functions, harmonic functions and integral operators.

In this paper, we investigate several results concerning the differential subordination and superordination of analytic functions in the open unit disc U , which are associated with derivative operator.

Let $H(U)$ denote the class of analytic functions in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $H[a, n]$ denote the subclass of $H(U)$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, with $H_0 = H[0, 1]$ and $H = H[1, 1]$.

If f, g are members of the analytic function class $H(U)$ we say that a function f is subordinate to a function g or g is said to be superordinate to f if there exists a function w with $w(0) = 0$, $|w(z)| < 1$ for all $(z \in U)$, such that $f(z) = g(w(z))$. In such a case we write $f \prec g$. Further, if the function g is univalent in U then we have the following equivalent $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let A denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (a \in \mathbb{C}, z \in U). \tag{1}$$

The author in [1] have recently introduced a new generalized derivative operator $D^{\alpha, n}(m, q, \lambda)f(z)$ as the following:

For the function $f \in A$ given by (1) we define a new generalized derivative operator $D^{\alpha, n}(m, q, \lambda)f(z) : A \rightarrow A$ as follows:

$$D^{\alpha, n}(m, q, \lambda)f(z) = z + \sum_{k=2}^{\infty} k^{\alpha} \left(1 + \frac{k-1}{1+q} \lambda\right)^m c(n, k) a_k z^k, \tag{2}$$

where $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $m \in \mathbb{Z}$, $\lambda > 0$, $q \geq 0$ and $c(n, k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $m = 0, 1, 2, \dots$, then

$$D^{\alpha, n}(m, q, \lambda)f(z) = \underbrace{\phi(z) * \dots * \phi(z)}_{(m)\text{-times}} * \left[\frac{z}{(1-z)^{n+1}} \right] * \sum_{k=1}^{\infty} k^{\alpha} z^k * f(z)$$

$$= R^n * D^\alpha(m, q, \lambda)f(z),$$

where $R^n = z + \sum_{k=2}^{\infty} c(n, k)z^k$, the Ruscheweyh derivative operator.

If $m = -1, -2, \dots$, then

$$D^{\alpha, n}(m, q, \lambda)f(z) = \underbrace{\phi(z) * \dots * \phi(z)}_{(-m)\text{-times}} * \left[\frac{z}{(1-z)^{n+1}} \right] * \sum_{k=1}^{\infty} k^\alpha z^k * f(z)$$

$$= R^n * D^\alpha(m, q, \lambda)f(z).$$

Note that:

$$D^{0,0}(0, q, \lambda)f(z) = D^{0,0}(1, 0, 0)f(z) = f(z), \quad \text{and}$$

$$D^{0,0}(1, q, \lambda)f(z) = zf'(z).$$

By specialising the parameters of $D^{\alpha, n}(m, q, \lambda)f(z)$, we get the following derivative and integral operators.

- The derivative operator introduced by Ruscheweyh [2];

$$D^{0, n}(0, q, \lambda) \equiv D^{0, n}(1, 0, 0); (n \in \mathbb{N}_0) \equiv R^n = z + \sum_{k=2}^{\infty} c(n, k)a_k z^k.$$

- The derivative operator introduced by Şahin [3];

$$D^{\alpha, 0}(0, q, \lambda) \equiv D_1^{\alpha, 0}(n, 0, 1); (n \in \mathbb{N}_0) \equiv D^n = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

- The generalised Salagean derivative operator introduced by Oboudi [4];

$$D^{0, 0}(n, 0, \lambda); (n \in \mathbb{N}_0) \equiv D_\lambda^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k.$$

- The generalised Ruscheweyh derivative operator introduced by Darus and Al-Shaqsi [5];

$$D^{0, n}(1, 0, \lambda); (n \in \mathbb{N}_0) \equiv R_\lambda^n = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))c(n, k)a_k z^k.$$

- The derivative operator introduced by Catas [6];

$$D^{0, \beta}(m, l, \lambda); (m \in \mathbb{N}_0) \equiv D^m(\lambda, \beta, l) = z + \sum_{k=2}^{\infty} \left(\frac{1 + \lambda(k-1) + l}{1+l} \right)^m c(\beta, k)a_k z^k.$$

- The integral operator introduced by Cho and T. H. Kim [7];

$$D^{1, 0}(-n, \lambda, 1) \equiv I_n^\lambda = z + \sum_{k=2}^{\infty} k \left(\frac{1 + \lambda}{k + \lambda} \right)^n a_k z^k.$$

It is easily verified from (2) that

$$(1+q)D^{\alpha, n}(m+1, q, \lambda)f(z) = \lambda z [D^{\alpha, n}(m, q, \lambda)f(z)]' + (1+q-\lambda)D^{\alpha, n}(m, q, \lambda)f(z). \quad (3)$$

Denote by Q the set of functions γ that are analytic and injective on $\bar{U} \setminus E(\gamma)$, where $E(\gamma) = \{\xi \in \partial U : \lim_{z \rightarrow \xi} \gamma(z) = \infty\}$,

and are such that $\gamma'(\xi) \neq 0$, $\xi \in \partial U \setminus E(\gamma)$. Further let the subclass of Q for which $\gamma(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

To prove our results, we need the following definitions and Lemmas.

Definition 1.1 [9] Let Ω be a set in \mathbb{C} ; $\gamma \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, \gamma]$ consists of those functions $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = \gamma(\xi)$, $s = k \xi \gamma'(\xi)$,

$$\Re \left\{ \frac{t}{s} + 1 \right\} \geq k \Re \left\{ 1 + \frac{\xi \gamma'(\xi)}{\gamma(\xi)} \right\},$$

where $z \in U$, $\xi \in \partial U \setminus E(\gamma)$ and $k \geq n$. We write $\Psi_1[\Omega, \gamma] = \Psi[\Omega, \gamma]$.

In particular when $\gamma(z) = M \frac{Mz + a}{M + \bar{a}z}$, with $M > 0$ and $|a| < M$, then $\gamma(U) = U_M = \{w : |w| < M\}$, $\gamma(0) = a$, $E(\gamma) = \emptyset$ and $\gamma \in Q$. In this case, we set $\Psi_n[\Omega, M, a] = \Psi_n[\Omega, \gamma]$, and in the special case when the set $Q = U_M$, the class is simply denoted by $\Psi_n[M, a]$.

Definition 1.2 [8] Let Ω be a set in \mathbb{C} , $\gamma(z) \in H[a, n]$ with $\gamma'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, \gamma]$ consists of those functions $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \xi) \in \Omega,$$

whenever $r = \gamma(z)$, $s = \frac{z \gamma'(z)}{\mu}$,

$$\Re \left\{ \frac{t}{s} + 1 \right\} \geq \frac{1}{\mu} \Re \left\{ 1 + \frac{\xi \gamma'(\xi)}{\gamma(\xi)} \right\},$$

where $z \in U$, $\xi \in \partial U$ and $\mu \geq k \geq 1$. In particular, we write $\Psi'_1[\Omega, \gamma] = \Psi'[\Omega, \gamma]$.

Lemma 1.3 [9] Let $\psi \in \Psi_n[\Omega, \gamma]$ with $\gamma(0) = a$. If the analytic function $g(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, satisfies

$$\psi(g(z), z g'(z), z^2 g''(z); z) \in \Omega,$$

then

$$g(z) \prec \gamma(z).$$

Lemma 1.4 [8] Let $\psi \in \Psi'_n[\Omega, \gamma]$ with $\gamma(0) = a$, $g \in Q(a)$ and

$$\psi(g(z), zg'(z), z^2g''(z), z)$$

is univalent in U , then

$$\Omega \subset \left\{ \psi(g(z), zg'(z), z^2g''(z), z), (z \in U) \right\}.$$

In the present investigation, the differential subordination result of Miller and Mocanu [8] is extended for functions associated with generalized derivative operator .

A similar problem for analytic functions was studied by many authors for example see ([10]-[11]). Additionally, the corresponding differential superordination problem is investigated, and several sandwich-type results are obtained.

2. Subordination Results Associated with Generalized Derivative operator

In this section, the differential subordination of generalized derivative operator is investigated. For this purpose, the class of admissible functions is given in the following definition.

Definition 2.1 Let Ω be a set in \mathbb{C} , $\gamma \in Q_0 \cap H[0,1]$. The class of admissible functions $\Phi_D[\Omega, \gamma]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = \gamma(\xi), \quad v = \frac{k\xi\gamma' + \frac{(1+q-\lambda)}{\lambda}g(z)}{\frac{1+q}{\lambda}},$$

$$\Re \left\{ \frac{\left(\frac{1+q}{\lambda} \right)^2 w - \left(\frac{1+q-\lambda}{\lambda} \right)^2 u}{\left(\frac{1+q}{\lambda} \right) v - \left(\frac{1+q-\lambda}{\lambda} \right) u} - 2 \left(\frac{1+q-\lambda}{\lambda} \right) \right\} \geq k \Re \left\{ 1 + \frac{\xi\gamma'(\xi)}{\gamma(\xi)} \right\},$$

where $z \in U$, $\xi \in \partial U \setminus E(\gamma)$, $\lambda > 0, q \geq 0$, and $k \geq 1$.

Theorem 2.2 Let $\phi \in \Psi_D[\Omega, \gamma]$. If $f \in A$ satisfies

$$\phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z)) \subset \Omega,$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$. Then

$$D^{\alpha,n}(m,q,\lambda)f(z) \prec \gamma(z), \quad (z \in U).$$

Proof: Define the analytic function g in U by

$$D^{\alpha,n}(m,q,\lambda)f(z) = g(z). \quad (\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U). \quad (4)$$

In view of the relation (3) from (4), we get

$$D^{\alpha,n}(m+1,q,\lambda)f(z) = \frac{zg'(z) + \frac{(1+q-\lambda)}{\lambda}g(z)}{\frac{1+q}{\lambda}}. \quad (5)$$

Further computations show that

$$D^{\alpha,n}(m+2,q,\lambda)f(z) = \frac{z^2g''(z) + (1+2\frac{(1+q-\lambda)}{\lambda})zg'(z) + (\frac{(1+q-\lambda)}{\lambda})^2g(z)}{(\frac{(1+q)}{\lambda})^2}. \quad (6)$$

Define the transformations from C^3 to C by

$$u = r, v = \frac{s + \frac{(1+q-\lambda)}{\lambda}r}{\frac{1+q}{\lambda}}, w = \frac{t + (1+2\frac{(1+q-\lambda)}{\lambda})s + (\frac{(1+q-\lambda)}{\lambda})^2r}{(\frac{(1+q)}{\lambda})^2}. \quad (7)$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) =$$

$$\phi\left(r, \frac{s + \frac{(1+q-\lambda)}{\lambda}r}{\frac{1+q}{\lambda}}, \frac{t + (1+2\frac{(1+q-\lambda)}{\lambda})s + (\frac{(1+q-\lambda)}{\lambda})^2r}{(\frac{(1+q)}{\lambda})^2}; z\right). \quad (8)$$

The proof shall make use of Lemma 1.3. Using equations (4), (5) and (6), from (8), we obtain $\psi(g(z), zg'(z), z^2g''(z); z) =$

$$\phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z); z), \quad (9)$$

where $\lambda > 0, q \geq 0, m, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

The proof is completed if it can be shown that the admissibility condition $\phi \in \Psi_D[\Omega, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{\left(\frac{1+q}{\lambda}\right)^2 w - \left(\frac{1+q-\lambda}{\lambda}\right)^2 u}{\frac{(1+q)}{\lambda} v - \frac{(1+q-\lambda)}{\lambda} u} - 2\left(\frac{1+q-\lambda}{\lambda}\right),$$

and hence $\psi \in \Psi[\Omega, \gamma]$, By Lemma 1.3,

$$g(z) \prec \gamma(z),$$

or $D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(z) (z \in U)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Psi_D[h(U), \gamma]$ is written as $\Psi_D[h, \gamma]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3 Let $\phi \in \Psi_D[h, \gamma]$. If $f \in A$ satisfies

$$\phi\left(D^{\alpha,n}(m, q, \lambda)f(z), D^{\alpha,n}(m+1, q, \lambda)f(z), D^{\alpha,n}(m+2, q, \lambda)f(z)\right) \prec h(z),$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$.

Then

$$D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(z), \quad (z \in U).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of $\gamma(z)$ on ∂U is not known.

Corollary 2.4 Let $\Omega \in \mathbb{C}$ and let γ be univalent in $U, \gamma(0) = 0$. Let $\Psi_D[\Omega, \gamma_\rho]$ for some $\rho \in (0, 1)$ where $\gamma_\rho(z) = \gamma(\rho z)$. If $f(z) \in A$ and

$$\phi\left(D^{\alpha,n}(m, q, \lambda)f(z), D^{\alpha,n}(m+1, q, \lambda)f(z), D^{\alpha,n}(m+2, q, \lambda)f(z)\right) \in \Omega,$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$.

Then

$$D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(z), \quad (z \in U).$$

Proof: Theorem 2.2 yields $D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(\rho z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

In the particular case $q(z) = Mz, M > 0$, and in view of the Definition 1.2, the class of admissible functions $\Psi_D[\Omega, \gamma]$ denoted by $\Psi_D[\Omega, M]$ is described below.

Definition 2.5 Let Ω be a set in \mathbb{C} , $M \geq 0$. The class of admissible functions $\Phi_D[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi \left(Me^{i\theta}, \frac{N + \frac{(1+q-\lambda)}{\lambda}}{\frac{1+q}{\lambda}} Me^{i\theta}, \frac{L + ((1+2\frac{(1+q-\lambda)}{\lambda})N + (\frac{1+q-\lambda}{\lambda})^2)Me^{i\theta}}{(\frac{1+q}{\lambda})^2}; z \right) \notin \Omega,$$

where $\lambda > 0, q \geq 0, \theta \in \mathbb{R}, \Re(Le^{i\theta}) \geq N(N-1)M$ for all real $\theta, N \geq 1, z \in U, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Corollary 2.6 Let $\phi \in \Psi_D[\Omega, \gamma]$. If $f \in A$ satisfies

$$\phi(D^{\alpha,n}(m, q, \lambda)f(z), D^{\alpha,n}(m+1, q, \lambda)f(z), D^{\alpha,n}(m+2, q, \lambda)f(z); z) \in \Omega,$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$. Then

$$|D^{\alpha,n}(m, q, \lambda)f(z)| < M, \quad (z \in U).$$

Proof: Theorem 2.2 gives

$$D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(z) = Mz,$$

$$D^{\alpha,n}(m, q, \lambda)f(z) \prec \gamma(z) = Mw(z).$$

Hence $|D^{\alpha,n}(m, q, \lambda)f(z)| < M$ where $|w(z)| < 1$.

In the special case $\Omega = \gamma(U) = \{w : |w| < M\}$ the class $\Psi_D[\Omega, M]$ is simply denoted by $\Psi_D[M]$.

Corollary 2.7 Let $\phi \in \Psi_D[\Omega, \gamma]$. If $f \in A$ satisfies

$$\left| \phi(D^{\alpha,n}(m, q, \lambda)f(z), D^{\alpha,n}(m+1, q, \lambda)f(z), D^{\alpha,n}(m+2, q, \lambda)f(z); z) \right| < M,$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$.

Then

$$|D^{\alpha,n}(m, q, \lambda)f(z)| < M, \quad (z \in U).$$

Definition 2.8 Let Ω be a set in \mathbb{C} ; $\gamma \in Q_1 \cap H$. The class of admissible functions $\Phi_{D,2}[\Omega, \gamma]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = \gamma(\xi), \quad v = \gamma(\xi) + \frac{\lambda}{1+q} \frac{k \xi \gamma'(\xi)}{\gamma(\xi)},$$

$$\Re \left\{ \frac{\left(\frac{1+q}{\lambda}\right)^2 w - \left(\frac{1+q-\lambda}{\lambda}\right)^2 u}{\frac{(1+q)}{\lambda} v - \frac{(1+q-\lambda)}{\lambda} u} - 2\left(\frac{1+q-\lambda}{\lambda}\right) \right\} \geq k \Re \left\{ 1 + \frac{\xi \gamma'(\xi)}{\gamma(\xi)} \right\},$$

where $z \in U$, $\xi \in \partial U \setminus E(\gamma)$, $\lambda > 0, q \geq 0$, and $k \geq 1$.

Theorem 2.9 Let $\phi \in \Psi_{D,2}[\Omega, \gamma]$. If $f \in A$ satisfies

$$\phi \left(\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+2, q, \lambda)f(z)}{D^{\alpha,n}(m+1, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+3, q, \lambda)f(z)}{D^{\alpha,n}(m+2, q, \lambda)f(z)}; z \right) \subset \Omega, \quad (10)$$

where $\lambda > 0, q \geq 0$, $m, \delta, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $z \in U$. Then

$$\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)} \prec \gamma(z), \quad (z \in U).$$

Proof: Define an analytic function g in U by

$$g(z) = \frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}. \quad (11)$$

Using (11), we get

$$\frac{zg'(z)}{g(z)} = \frac{z(D^{\alpha,n}(m+1, q, \lambda)f(z))'}{D^{\alpha,n}(m+1, q, \lambda)f(z)} - \frac{z(D^{\alpha,n}(m, q, \lambda)f(z))'}{D^{\alpha,n}(m, q, \lambda)f(z)}. \quad (12)$$

By making use of (2) in (12), we get

$$\frac{D^{\alpha,n}(m+2, q, \lambda)f(z)}{D^{\alpha,n}(m+1, q, \lambda)f(z)} = g(z) + \left(\frac{\lambda}{1+q} \right) \frac{zg'(z)}{g(z)}. \quad (13)$$

Further computations show that

$$\frac{D^{\alpha,n}(m+3, q, \lambda)f(z)}{D^{\alpha,n}(m+2, q, \lambda)f(z)} = \frac{1}{\frac{zg'}{\left(\frac{1+q}{\lambda}\right)g(z)} + g(z)} \left[\frac{z^2 g''(z)}{\left(\frac{1+q}{\lambda}\right)^2 g(z)} + \frac{zg'(z)}{\left(\frac{1+q}{\lambda}\right)^2 g(z)} + \frac{3zg'(z)}{\left(\frac{1+q}{\lambda}\right)} + g(z) \right]. \quad (14)$$

Define the transformation from C^3 to C by

$$u = r, \quad v = r + \frac{s}{\left(\frac{1+q}{\lambda}\right)r},$$

$$w = \frac{1}{\frac{s}{\frac{1+q}{\lambda}r} + r} \left[\frac{t}{\left(\frac{1+q}{\lambda}\right)^2 r} + \frac{s}{\left(\frac{1+q}{\lambda}\right)^2 r} + \frac{3s}{\left(\frac{1+q}{\lambda}\right)} + r \right].$$

Let

$$\psi(r, s, t; z) = \phi(u, v, w; z) = \phi \left(r, r + \frac{s}{\frac{1+q}{\lambda}r}, \frac{1}{\frac{s}{\frac{p+q}{\lambda}r} + r} \left[\frac{t}{\left(\frac{1+q}{\lambda}\right)^2 r} + \frac{s}{\left(\frac{1+q}{\lambda}\right)^2 r} + \frac{3s}{\left(\frac{1+q}{\lambda}\right)} + r \right]; z \right). \quad (15)$$

The proof shall make use of Lemma 1.3. Using equations (12), (13) and (14), from (15), we obtain

$$\psi(g(z), zg'(z), z^2 g''(z); z) = \phi \left(\frac{D^{\alpha, n}(m+1, q, \lambda)f(z)}{D^{\alpha, n}(m, q, \lambda)f(z)}, \frac{D^{\alpha, n}(m+2, q, \lambda)f(z)}{D^{\alpha, n}(m+1, q, \lambda)f(z)}, \frac{D^{\alpha, n}(m+3, q, \lambda)f(z)}{D^{\alpha, n}(m+2, q, \lambda)f(z)}; z \right), \quad (16)$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$.

Hence (10) becomes

$$\psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Psi_{D,2}[\Omega, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \left(\frac{(1+q)v}{\lambda(u-v)} \right) w - \frac{(3v-2u)\frac{p+q}{\lambda}}{u-v},$$

and hence $\psi \in \Psi_D[\Omega, \gamma]$. By Lemma 1.3,

$$g(z) \prec \gamma(z) \quad \text{or} \quad \frac{D^{\alpha, n}(m+1, q, \lambda)f(z)}{D^{\alpha, n}(m, q, \lambda)f(z)} \prec \gamma(z), \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Psi_{D,2}[h(U), \gamma]$.

In the particular case $\gamma(z) = Mz, M > 0$, and in view of the definition 1.2, the class of admissible functions $\Psi_{D,2}[\Omega, \gamma]$ denoted by $\Psi_{D,2}[\Omega, M]$ is described below.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.9.

Theorem 2.10 Let $\phi \in \Psi_{D,2}[h, \gamma]$. If $f \in A$ satisfies

$$\phi \left(\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+2,q,\lambda)f(z)}{D^{\alpha,n}(m+1,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+3,q,\lambda)f(z)}{D^{\alpha,n}(m+2,q,\lambda)f(z)}; z \right) \prec h(z),$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$. Then

$$\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)} \prec \gamma(z), \quad (z \in U).$$

Definition 2.11 Let Ω be a set in \mathbb{C} , $M \geq 0$. The class of admissible functions $\Phi_{D,2}[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; z) = \phi \left(Me^{i\theta}, Me^{i\theta} + \frac{N}{1+q}, \frac{1}{M^2 e^{i\theta} + \frac{(1+q)MN}{\lambda}} \left[M^3 e^{2i\theta} + \frac{3M^2 Ne^{i\theta}}{1+q} + \frac{Le^{-i\theta} + MN}{\left(\frac{1+q}{\lambda}\right)^2} \right]; z \right),$$

where $\lambda > 0, q \geq 0, \theta \in \mathbb{R}, \Re(Le^{i\theta}) \geq N(N-1)M, N \geq 1$ for all real θ .

Corollary 2.12 Let $\phi \in \Psi_{D,2}[\Omega, \gamma]$. If $f \in A$ satisfies

$$\phi \left(\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+2,q,\lambda)f(z)}{D^{\alpha,n}(m+1,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+3,q,\lambda)f(z)}{D^{\alpha,n}(m+2,q,\lambda)f(z)}; z \right) \in \Omega,$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$.

$$\text{Then } \left| \frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)} \right| < M, \quad (z \in U).$$

3. Superordination Results Associated with Generalized Derivative operator

Definition 3.1 Let Ω be a set in \mathbb{C} ; $\gamma \in Q_0 \cap H[0, 1], z\gamma'(z) \neq 0$. The class of admissible functions $\Phi'_D[\Omega, \gamma]$ consists of those functions $\phi: \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi) \notin \Omega,$$

whenever

$$u = \gamma(\xi), \quad v = \frac{z\gamma' + \mu \frac{(1+q-\lambda)}{\lambda} \gamma(z)}{\mu \left(\frac{1+q}{\lambda}\right)},$$

$$\Re \left\{ \frac{\left(\frac{1+q}{\lambda}\right)^2 w - \left(\frac{1+q-\lambda}{\lambda}\right)^2 u}{\left(\frac{1+q}{\lambda}\right) v - \frac{(1+q-\lambda)}{\lambda} u} - 2 \left(\frac{1+q-\lambda}{\lambda}\right) \right\} \geq \frac{1}{\mu} \Re \left\{ 1 + \frac{z\gamma'(z)}{\gamma(z)} \right\},$$

where $z \in U$, $\xi \in \partial U \setminus E(\gamma)$, $\lambda > 0, q \geq 0$, and $\mu \geq 1$.

Theorem 3.2 Let $\phi \in \Psi'_D[\Omega, \gamma]$. If $f \in A$, $D^{\alpha, n}(m, q, \lambda)f(z) \in H_0$ and

$$\phi\left(D^{\alpha, n}(m, q, \lambda)f(z), D^{\alpha, n}(m+1, q, \lambda)f(z), D^{\alpha, n}(m+2, q, \lambda)f(z)\right)$$

is univalent in U , then

$$\Omega \subset \phi\left(D^{\alpha, n}(m, q, \lambda)f(z), D^{\alpha, n}(m+1, q, \lambda)f(z), D^{\alpha, n}(m+2, q, \lambda)f(z)\right), \quad (17)$$

where $\lambda > 0, q \geq 0$, $m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $z \in U$,

implies $\gamma(z) \prec D^{\alpha, n}(m, q, \lambda)f(z)$, ($z \in U$).

Proof: From (9) and (17), we have

$$\Omega \subset \phi\left(D^{\alpha, n}(m, q, \lambda)f(z), D^{\alpha, n}(m+1, q, \lambda)f(z), D^{\alpha, n}(m+2, q, \lambda)f(z)\right), \quad (z \in U)$$

From (7), we see that the admissibility condition for $\phi \in \Psi'_D[\Omega, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence and by Lemma 1.4, we get $\gamma(z) \prec g(z)$.

$$\gamma(z) \prec D^{\alpha, n}(m, q, \lambda)f(z), \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Psi'_D[h(U), \gamma]$ is written as $\Psi'_Q[h, \gamma]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.2.

Theorem 3.3 Let $h(z)$ is analytic on U and $\phi \in \Psi'_D[h, \gamma]$. If $f \in A$ $D^{\alpha, n}(m, q, \lambda)f(z) \in H_0$ and

$$\phi\left(D^{\alpha, n}(m, q, \lambda)f(z), D^{\alpha, n}(m+1, q, \lambda)f(z), D^{\alpha, n}(m+2, q, \lambda)f(z)\right),$$

is univalent in U , then

$$h(z) \prec \phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z)), \quad (18)$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$,

implies

$$\gamma(z) \prec D^{\alpha,n}(m,q,\lambda)f(z), \quad (z \in U).$$

Proof: From (18) we get

$$h(z) = \Omega \subset \phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z)),$$

and so by Theorem 3.2, we get

$$\gamma(z) \prec D^{\alpha,n}(m,q,\lambda)f(z), \quad (z \in U).$$

Combining Theorems 2.3 and 3.3, we obtain the following Sandwich-type theorem.

Corollary 3.4 Let $h_1(z)$ and $\gamma_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $\gamma_2(z) \in H_0$ with $\gamma_1(0) = \gamma_2(0) = 0$ and $\phi \in \Psi_D[h_1, \gamma_1] \cap \Psi_D[h_2, \gamma_2]$. If $f \in A$, $D^{\alpha,n}(m,q,\lambda)f(z) \in H[0,1] \cap H_0$ and

$$\phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z)),$$

is univalent in U , then

$$h_1(z) \prec \phi(D^{\alpha,n}(m,q,\lambda)f(z), D^{\alpha,n}(m+1,q,\lambda)f(z), D^{\alpha,n}(m+2,q,\lambda)f(z)) \prec h_2(z),$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$,

implies $\gamma_1(z) \prec D^{\alpha,n}(m,q,\lambda)f(z) \prec \gamma_1(z), \quad (z \in U)$.

Definition 3.5 Let Ω be a set in \mathbb{C} , $\gamma \in Q_1 \cap H$. The class of admissible functions $\Phi_{D,2}[\Omega, \gamma]$ consists of those functions $\phi: \mathbb{C}^3 \times \bar{U} \rightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \xi) \notin \Omega,$$

whenever

$$u = q(z), \quad v = g(z) + \frac{\lambda}{m(p+q)} \frac{zg(z)}{g(z)},$$

$$\Re \left\{ \frac{\left(\frac{1+q}{\lambda} \right)^2 w - \left(\frac{1+q-\lambda}{\lambda} \right)^2 u}{\left(\frac{1+q}{\lambda} \right) v - \left(\frac{1+q-\lambda}{\lambda} \right) u} - 2 \left(\frac{1+q-\lambda}{\lambda} \right) \right\} \geq \frac{1}{m} \Re \left\{ 1 + \frac{zq'(z)}{q(z)} \right\},$$

where $z \in U$, $\xi \in \partial U \setminus E(\gamma)$, $\lambda > 0, q \geq 0$, and $m \geq 1$.

Now we will give the dual result of Theorem 2.9 for the differential superordination.

Theorem 3.6 Let $\phi \in \Psi'_{1,2}[\Omega, q]$. If $f \in A$, $\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)} \in H_0$, and

$$\phi \left(\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+2, q, \lambda)f(z)}{D^{\alpha,n}(m+1, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+3, q, \lambda)f(z)}{D^{\alpha,n}(m+2, q, \lambda)f(z)} \right),$$

where $\lambda > 0, q \geq 0$, $m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $z \in U$, is univalent in U , then

$$\Omega \subset \phi \left(\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+2, q, \lambda)f(z)}{D^{\alpha,n}(m+1, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+3, q, \lambda)f(z)}{D^{\alpha,n}(m+2, q, \lambda)f(z)} \right), \quad (19)$$

implies

$$\gamma(z) \prec \frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \quad (z \in U).$$

Proof: From (15) and (19), we have

$$\Omega \subset \psi(g(z), zg'(z), z^2g''(z); z).$$

From (16), we see that the admissibility condition for $\phi \in \Psi'_{D,2}[\Omega, \gamma]$ is equivalent to the admissibility condition for Ψ as given in Definition 1.2.

Hence $\psi \in \Psi'_{D,2}[\Omega, \gamma]$ and by Lemma 1.4, we get $\gamma(z) \prec g(z)$

$$\gamma(z) \prec \frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \quad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω . In this case the class $\Psi'_{D,2}[h(U), \gamma]$, is written as $\Psi'_{D,2}[h, \gamma]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.8.

Theorem 3.7 Let $\gamma \in H$, and let h be analytic on U , and $\phi \in \Psi'_{D,2}[\Omega, \gamma]$. If

$$f \in A, \frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)} \in H_1, \text{ and}$$

$$\phi \left(\frac{D^{\alpha,n}(m+1, q, \lambda)f(z)}{D^{\alpha,n}(m, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+2, q, \lambda)f(z)}{D^{\alpha,n}(m+1, q, \lambda)f(z)}, \frac{D^{\alpha,n}(m+3, q, \lambda)f(z)}{D^{\alpha,n}(m+2, q, \lambda)f(z)} \right)$$

where $\lambda, m, q \geq 0$, $n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $z \in U$, is univalent in U , then

$$h(z) \prec \phi \left(\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+2,q,\lambda)f(z)}{D^{\alpha,n}(m+1,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+3,q,\lambda)f(z)}{D^{\alpha,n}(m+2,q,\lambda)f(z)} \right),$$

implies

$$h(z) \prec \frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \quad (z \in U).$$

Combining Theorem 2.10 and 3.7, we obtain the following Sandwich-type theorem.

Corollary 3.8 Let $h_1(z)$ and $\gamma_1(z)$ be analytic functions in U , $h_2(z)$ be univalent function in U , $\gamma_2(z) \in H_0$ with $\gamma_1(0) = \gamma_2(0) = 0$, and $\phi \in \Psi_{D,2}[h_1, h_1] \cap \Psi_{D,2}[h_2, q_2]$. If $f \in A$, $D^{\alpha,n}(m+1,q,\lambda)f(z) \in H[0,1] \cap H_0$, and

$$\phi \left(\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+2,q,\lambda)f(z)}{D^{\alpha,n}(m+1,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+3,q,\lambda)f(z)}{D^{\alpha,n}(m+2,q,\lambda)f(z)} \right),$$

is univalent in U , then

$$h_1(z) \prec \phi \left(\frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+2,q,\lambda)f(z)}{D^{\alpha,n}(m+1,q,\lambda)f(z)}, \frac{D^{\alpha,n}(m+3,q,\lambda)f(z)}{D^{\alpha,n}(m+2,q,\lambda)f(z)} \right) \prec h_2(z), \quad (30)$$

where $\lambda > 0, q \geq 0, m, n, \alpha \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, z \in U$, implies

$$\gamma_1(z) \prec \frac{D^{\alpha,n}(m+1,q,\lambda)f(z)}{D^{\alpha,n}(m,q,\lambda)f(z)} \prec \gamma_2(z), \quad (z \in U).$$



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