Tamsui Oxford Journal of Information and Mathematical Sciences **28(4)** (2012) 379-393
Aletheia University

Inclusion Relations for Subclasses of Analytic Functions Defined by Integral Operator Associated with the Hurwitz-Lerch Zeta Function *

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Received March 29, 2011, Accepted March 13, 2012.

Abstract

The main objective of this article is to introduce a new integral operator $\Im_{s,b}^{\alpha} f(z)$ defined by using the fractional derivative for Hurwitz. Lerch zeta function. This operator was motivated by many researchers namely Srivastava, Srivastava and Attiya, and many others. Inclusion relations for new subclasses of analytic functions defined by operator aforementioned are also considered.

Keywords and Phrases: *Fractional derivative, Hurwitz-Lerch zeta functions, Inclusion relations.*

1. Introduction

Let $\mathbb{A}$ denote the class of all analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, given by the normalized power series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

*2000 *Mathematics Subject Classification.* Primary 30C45.

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For two analytic functions $f(z)$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their convolution (or Hadamard product) is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

A function $f \in \mathbb{A}$ is said to be in the class denoted by $SP(k, \beta)$, $(-1 \leq \beta < 1)$, and satisfies

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta, \quad (k \geq 0, \quad and \quad \beta + k \geq 0, \quad z \in \mathbb{U}). \tag{1.2}$$

Furthermore, a function $f \in \mathbb{A}$ is said to be in the class $UC(k, \beta)$ of k-uniformly convex of order $\beta \quad (-1 \leq \beta < 1)$, and satisfies

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k\left|1 + \frac{zf''(z)}{f'(z)}\right| + \beta, \quad (k \geq 0, \quad and \quad \beta + k \geq 0, z \in \mathbb{U}). \tag{1.3}$$

It is obvious that $f \in UC(k, \beta)$ if and only if $zf' \in SP(k, \beta)$. These classes generalize various other classes. For $k = 0$, we get, the classes $CV(\beta)$ and $ST(\beta)$ which are convex functions of order β and starlike functions of order β respectively.

The class $UC(1, 0) \equiv UC$ is called uniformly convex introduced by Goodman with geometric interpretation in [6]. The class $SP(1, 0) \equiv SP$ is defined by Ronning in [14]. The classes $UC(1, \beta) \equiv UC(\beta)$ and $SP(1, \beta) \equiv SP(\beta)$ are investigated by Ronning in [13]. For $\beta = 0$, the classes $UC(k, 0) \equiv k - UC$ and $SP(k, 0) \equiv k - SP$, respectively, are defined by Kanas and Wisniowska in [8] and [9].

Geometric interpretation. Let $f \in SP(k, \beta)$ and $f \in UC(k, \beta)$ if and only if $\frac{zf'(z)}{f(z)}$ and $\frac{zf''(z)}{f'(z)} + 1$ respectively, take all the values in the conic domain $R_{k,\beta}$ which is included in the right half plane such that

$$R_{k,\beta} = \{u + iv : u > k\sqrt{(u-1)^2 + v^2} + \beta\}, \tag{1.4}$$

with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = \frac{zf''(z)}{f'(z)} + 1$ and the considering the functions which map $\mathbb{U}$ onto the conic domain $R_{k,\beta}$, such that $1 \in R_{k,\beta}$, we may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) \prec q_{k,\beta}(z). \tag{1.5}$$

The functions that play the role of extremal functions for these classes in the case $k = 0$ and $k = 1$ can be found in [2] as follows:

$$\begin{aligned} q_{0,\beta} &= \frac{1+(1-2\beta)z}{1-z}, \\ q_{1,\beta} &= 1+\frac{2(1-\beta)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right). \end{aligned}$$

From (1.4) and the properties of the domains $R_{k,\beta}$ we have

$$\Re(p(z)) > \Re(q_{k,\beta}) > \frac{k+\beta}{k+1}. \tag{1.6}$$

Define $UCC(k,\gamma,\beta)$ to be the family of functions $f \in \mathbb{A}$ such that

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} \geq k\left|\frac{zf'(z)}{g(z)}-1\right|+\gamma, \quad (k \geq 0, \quad and \quad \beta+k \geq 0, \quad z \in \mathbb{U}), \tag{1.7}$$

for some $g \in SP(k,\beta)$.

Similarly, define $UQC(k,\gamma,\beta)$ to be the family of functions $f \in \mathbb{A}$ such that

$$Re\left\{\frac{(zf'(z))'}{g'(z)}\right\} \geq k\left|\frac{(zf'(z))'}{g'(z)}-1\right|+\gamma, \quad (k \geq 0, \quad and \quad \beta+k \geq 0, \quad z \in \mathbb{U}), \tag{1.8}$$

for some $g \in SP(k,\beta)$,

if $UCC(0,\gamma,\beta)$ is the class of close to convex functions of order γ and type β and $UQC(0,\gamma,\beta)$ is the class of quasi convex functions of order γ and type β.

Let

$$\varphi(a,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1}, \quad (z \in \mathbb{U}, c \neq 0,-1,-2,-3,...),$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(x)_k = \begin{cases} 1 \quad for \quad k=0, \\ x(x+1)(x+2)...(x+k-1) \quad for \quad k \in N = \{1,2,3,...\}. \end{cases}$

Carlson and Shaffer [4] introduced a linear operator $L(a,c)$ by

$$L(a,c)f(z) = \varphi(a,c;z) * f(z),$$

Note that:

$L(a,a)$ is the identity operator,and $L(a,c) = L(a,b)L(b,c) \quad (b,c \neq 0,-1,...)$.

In order to introduce a new integral operator we need the following definitions.

Definition 1.1. (Srivastava and Choi [17]) A general Hurwitz Lerch Zeta function $\Phi(z,s,b)$ defined by

$$\Phi(z,s,b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s},$$

where $s \in \mathbb{C}$, $b \in \mathbb{C} - \mathbb{Z}_0 \quad when \quad |z| < 1, \quad \Re(b) > 1 \quad when \quad |z| = 1$.

We define the function :

$$\Phi^*(z,s,b) = (b^s z \Phi(z,s,b)) * f(z),$$

then

$$\Phi^*(z,s,b) = z + \sum_{n=2}^{\infty} \frac{b^s}{(n+b-1)^s} a_n z^n.$$

Definition 1.2. (see [12] ,[16]) Let the function f be analytic in a simply connected domain of the z-plane containing the origin. The fractional derivative of f of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \qquad (0 \leq \alpha < 1),$$

where the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $log(z-t)$ to be real when $z - t > 0$.

Using Definition 1.2 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [12] introduced the operator $\Omega^{\alpha} : \mathbb{A} \to \mathbb{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^{\alpha} f(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \qquad (\alpha \neq 2,3,4,\cdots)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n \quad z \in \mathbb{U}.$$

Now, by using Definition 1.1, and Definition 1.2, the authors [18] have recently introduced a new generalized integral operator. For $s \in \mathbb{C}$, $b \in \mathbb{C} - \mathbb{Z}_0^-$, and $(\Im_{s,b}^{\alpha} f) : \mathbb{A} \to \mathbb{A}$ as the following:

$$\begin{aligned}
\Im_{s,b}^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z,s,b), \quad (\alpha \neq 2,3,4,\cdots) \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} . (\frac{b}{n-1+b})^s a_n z^n. \qquad (1.9) \\
&= z + \sum_{n=2}^{\infty} \varphi(2, 2-\alpha; z) * \psi(z,s,b), \\
\textit{where} \qquad \psi(z,s,b) &= z + \sum_{n=2}^{\infty} (\frac{b}{n-1+b})^s a_n z^n, \\
&= L(2, 2-\alpha) \psi(z, s.b) \\
&= \Omega^{\alpha} \psi(z, s.b),
\end{aligned}$$

then

$$\Im_{s,b}^{\alpha} f(z) = \Omega^{\alpha} \psi(z, s.b).$$

Note that :

$\Im_{0,b}^{0} f(z) = f(z)$.

Special cases of this operator includes:

- $\Im_{0,b}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z)$ is Owa and Srivastava operator [12].
- $\Im_{s,b+1}^{0} f(z) \equiv J_{s,b} f(z)$ is the Srivastava and Attiya integral operator [15].
- $\Im_{1,1}^{0} f(z) \equiv A(f)(z)$ is the Alexander integral operators [1].
- $\Im_{s+1,1}^{0} f(z) \equiv L(f)(z)$ is the Libera integral operators [10].
- $\Im_{1,\delta}^{0} f(z) \equiv L_{\delta}(f)(z)$ is the Bernardi integral operator [3].
- $\Im_{\sigma,2}^{0} f(z) \equiv I^{\sigma} f(z)$ is the Jung-Kim-Srivastava integral operator [7].

It is easily verified from the above definition of the operator $\Im_{s,b}^{\alpha} f(z)$ that:

$$z(\Im_{s+1,b}^{\alpha} f(z))' = (1-b)\Im_{s+1,b}^{\alpha} f(z) + b \Im_{s,b}^{\alpha} f(z). \qquad (1.10)$$

By using our integral operator we introduce the following subclasses of $\mathbb{A}$

Definition 1.3. Let $f(z) \in \mathbb{A}$. Then $f(z) \in SP^{\alpha}_{s,b}(k,\beta)$ if and only if $\Im^{\alpha}_{s,b} \in SP(k,\beta)$.

Definition 1.4. Let $f(z) \in \mathbb{A}$. Then $f(z) \in UC^{\alpha}_{s,b}(k,\beta)$ if and only if $\Im^{\alpha}_{s,b} \in UC(k,\beta)$.

Definition 1.5. Let $f(z) \in \mathbb{A}$. Then $f(z) \in UCC^{\alpha}_{s,b}(k,\gamma,\beta)$ if and only if $\Im^{\alpha}_{s,b} \in UCC(k,\gamma,\beta)$.

Definition 1.6. Let $f(z) \in \mathbb{A}$. Then $f(z) \in UQC^{\alpha}_{s,b}(k,\gamma,\beta)$ if and only if $\Im^{\alpha}_{s,b} \in UQC(k,\gamma,\beta)$.

We note that

$$f(z) \in UC^{\alpha}_{s,b}(k,\beta) \Leftrightarrow zf'(z) \in SP^{\alpha}_{s,b}(k,\beta). \tag{1.11}$$

2. Preliminaries Results

We need the following lemmas in our investigation

Lemma 2.1. *(Eenigenburg, Miller, Mocanu, and Read[5]) Let a, b be complex number and let h be convex univalent in unit disk $\mathbb{U}$ with $h(0) = c$ and $\Re\{ah(z) + b\} > 0$. Let $g(z) = c + \sum_{n=1}^{\infty} p_n z^n$ be analytic in $\mathbb{U}$. Then*

$$g(z) + \frac{zg'(z)}{ag(z) + b} \prec h(z), \quad (z \in \mathbb{U}),$$

implies

$$g(z) \prec h(z).$$

Lemma 2.2. *(see Miller and Mocanu[11]) Let h be convex in the unit disk $\mathbb{U}$ and let $A > 0$. Suppose $B(z)$ is analytic in $\mathbb{U}$ with with $\Re\{B(z)\} \geq A$. If g is analytic in $\mathbb{U}$ and $g(0) = h(0)$. Then*

$$Az^2 g''(z) + B(z)zg'(z) + g(z) \prec h(z) \Rightarrow g(z) \prec h(z).$$

3. Inclusion Relations

In the following results we will study inclusion relations

Theorem 3.1. *Let* $\Re(b) > \frac{1-\beta}{k+1}$, *and* $f \in \mathbb{A}$.

$$SP^{\alpha}_{s,b}(k,\beta) \subset SP^{\alpha}_{s+1,b}(k,\beta).$$

Proof. Let $f \in SP^{\alpha}_{s,b}(k,\beta)$. Then upon setting

$$\frac{z(\Im^{\alpha}_{s+1,b}f(z))'}{\Im^{\alpha}_{s+1,b}f)(z)} = p(z), \qquad (z \in \mathbb{U}), \tag{3.1}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$, is analytic in $\mathbb{U}$, with p(0)=1 and $p(z) \neq 0$ for all $z \in \mathbb{U}$.

From (1.10) we can write

$$b\frac{\Im^{\alpha}_{s,b}f(z)}{\Im^{\alpha}_{s+1,b}f)(z)} = \frac{z(\Im^{\alpha}_{s+1,b}f(z))'}{\Im^{\alpha}_{s+1,b}f)(z)} - (1-b), \tag{3.2}$$

$$b\frac{\Im^{\alpha}_{s,b}f(z)}{\Im^{\alpha}_{s+1,b}f)(z)} = p(z) - (1-b). \tag{3.3}$$

By logarithmically differentiating both sides of the equation (3.3), we get

$$\frac{z(\Im^{\alpha}_{s,b}f(z))'}{(\Im^{\alpha}_{s,b}f)(z)} = \frac{z(\Im^{\alpha}_{s+1,b}f(z))'}{\Im^{\alpha}_{s+1,b}f(z)} + \frac{zp'(z)}{p(z)-(1-b)},$$

$$\frac{z(\Im^{\alpha}_{s,b}f(z))'}{(\Im^{\alpha}_{s,b}f)(z)} = p(z) + \frac{zp'(z)}{p(z)+(b-1)}.$$

From this equation and the argument given in (1.5), we may write

$$p(z) + \frac{zp'(z)}{p(z)+(b-1)} \prec q_{k,\beta}(z).$$

Therefore, the theorem follows by Lemma 2.1, and the condition (1.5), since $q_{k,\beta}$ is univalent and convex in $\mathbb{U}$ and $\Re(q_{k,\beta}) > \frac{k+\beta}{k+1}$ that is, $f \in SP^{\alpha}_{s+1,b}(k,\beta)$.

Theorem 3.2. *Let* $\Re(b) > \frac{1-\beta}{k+1}$, *and* $f \in \mathbb{A}$, *then* $UC^{\alpha}_{s,b}(\gamma) \subset UC^{\alpha}_{s+1,b}(k,\beta)$.

Proof. Applying (1.2),(1.3) and Theorem 3.1, we observe that

$$\begin{aligned}
f(z) \in UC^{\alpha}_{s,b}(k,\beta) &\Leftrightarrow (\Im^{\alpha}_{s,b}f)(z) \in UC(k,\beta)\\
&\Leftrightarrow z(\Im^{\alpha}_{s,b}f(z))' \in SP(k,\beta)\\
&\Leftrightarrow \Im^{\alpha}_{s,b}(zf'(z)) \in SP(k,\beta)\\
&\Leftrightarrow zf'(z) \in SP^{\alpha}_{s,b}(k,\beta)\\
&\Rightarrow zf'(z) \in SP^{\alpha}_{s+1,b}(k,\beta)\\
&\Leftrightarrow \Im^{\alpha}_{s+1,b}z(f(z))' \in SP(k,\beta)\\
&\Leftrightarrow z(\Im^{\alpha}_{s+1,b}f(z))' \in SP(k,\beta)\\
&\Leftrightarrow \Im^{\alpha}_{s+1,b}f(z) \in UC(k,\beta)\\
&\Leftrightarrow f(z) \in UC^{\alpha}_{s+1,b}(k,\beta),
\end{aligned}$$

the proof is complete .

Theorem 3.3. *Let* $\Re(b) > \frac{1-\beta}{k+1}$, *and* $f \in \mathbb{A}$, *then*

$$UCC^{\alpha}_{s,b}(k,\beta) \subset UCC^{\alpha}_{s+1,b}(k,\beta).$$

Proof. Let $f(z) \in UCC^{\alpha}_{s,b}(k,\beta)$. Then, in view of the definition ,we can write

$$Re\left\{\frac{z(\Im^{\alpha}_{s,b}f(z))'}{\psi(z)}\right\} \prec q_{k,\beta} \quad (z \in \mathbb{U}),$$

for some $\psi(z) \in SP(k,\beta)$. Choose the function $g(z)$ such that $\Im^{\alpha}_{s,b}g(z) = \psi(z)$, so we have

$$Re\frac{z(\Im^{\alpha}_{s,b}f(z))'}{\Im^{\alpha}_{s,b}g(z)} \prec q_{k,\beta}. \tag{3.4}$$

New we set

$$\frac{z(\Im^{\alpha}_{s+1,b}f(z))'}{\Im^{\alpha}_{s+1,b}g(z)} = p(z), \tag{3.5}$$

where $p(z) = 1 + p_1z + p_2z^2 + \ldots$, is analytic in $\mathbb{U}$,$p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$.

Using the identity (1.10) we have we have

$$\begin{aligned}\frac{z(\Im^{\alpha}_{s,b}f(z))'}{\Im^{\alpha}_{s,b}g(z)} &= \frac{\Im^{\alpha}_{s,b}(zf'(z))}{\Im^{\alpha}_{s,b}g(z)},\\ &= \frac{z(\Im^{\alpha}_{s+1,b}zf'(z))'-(1-b)(\Im^{\alpha}_{s+1,b}zf'(z))}{z(\Im^{\alpha}_{s+1,b}g(z))'-(1-b)(\Im^{\alpha}_{s+1,b}g(z))}\\ &= \frac{\frac{z(\Im^{\alpha}_{s+1,b}zf'(z))'}{\Im^{\alpha}_{s+1,b}g(z)}-\frac{(1-b)(\Im^{\alpha}_{s+1,b}zf'(z))}{\Im^{\alpha}_{s+1,b}g(z)}}{\frac{z(\Im^{\alpha}_{s+1,b}g(z)}{\Im^{\alpha}_{s+1,b}g(z)}-(1-b)}. \qquad (3.6)\end{aligned}$$

Since $g(z)\in SP^{\alpha}_{s+1,b}(k,\beta)$, and by Theorem 3.1, we can write $\frac{z(\Im^{\alpha}_{s+1,b}g(z))'}{\Im^{\alpha}_{s+1,b}g(z)} = r(z)$, where $\Re\{r(z)\}>0$, $(z\in U)$,

$$\frac{z(\Im^{\alpha}_{s,b}f(z))'}{\Im^{\alpha}_{s,b}g(z)} = \frac{\frac{z(\Im^{\alpha}_{s+1,b}zf'(z))'}{\Im^{\alpha}_{s+1,b}g(z)}-(1-b)p(z)}{r(z)-(1-b)}. \qquad (3.7)$$

From (3.5) we consider that

$$z(\Im^{\alpha}_{s+1,b}f(z))' = \Im^{\alpha}_{s+1,b}g(z)[p(z)] \qquad (3.8)$$

differentiating both sides of (3.8) with respect to z, we get

$$\frac{z[z(\Im^{\alpha}_{s+1,b}f(z))']'}{\Im^{\alpha}_{s+1,b}g(z)} = zp'(z)+p(z)(r(z)). \qquad (3.9)$$

Using (3.8) and (3.9), we obtain

$$\begin{aligned}\frac{z(\Im^{\alpha}_{s,b}f(z))'}{\Im^{\alpha}_{s,b}g(z)} &= \frac{p(z).r(z)+zp'(z)-(1-b)p(z)}{r(z)-(1-b)},\\ &= p(z)+\frac{zp'(z)}{r(z)-(1-b)}. \qquad (3.10)\end{aligned}$$

From (3.4) and (3.10) we conclude that

$$p(z)+\frac{1}{r(z)-(1-b)}zp'(z)\prec q_{k,\beta}.$$

For letting $A=0$ and $B(z)=\frac{1}{r(z)-(1-b)}$,we obtain

$$\Re\{B(z)\}=\frac{1}{r(z)+(b-1)}=\frac{1}{(r(z)+(b-1))^2}\Re[r(z)+(b-1)]>0.$$

The above inequality satisfies the conditions required by Lemma 2.2 . Hence $p(z)\prec q_{k,\beta}$

so the proof is complete.

Theorem 3.4. *Let $\Re b>\frac{1-\beta}{k+1}$, and $f\in\mathbb{A}$, then*

$$UQC^{\alpha}_{s,b}(k,\beta)\subset UQC^{\alpha}_{s+1,b}(k,\beta). \tag{3.11}$$

Proof. Applying (1.7),(1.8) and Theorem 3.3, we observe that

$$\begin{aligned}
f(z)\in UQC^{\alpha}_{s,b}(k,\beta) \quad &\Leftrightarrow \quad (\Im^{\alpha}_{s,b}f)(z)\in UQC(k,\beta)\\
&\Leftrightarrow \quad z(\Im^{\alpha}_{s,b}f(z))'\in UCC(k,\beta)\\
&\Leftrightarrow \quad \Im^{\alpha}_{s,b}(zf'(z))\in UCC(k,\beta)\\
&\Leftrightarrow \quad zf'(z)\in UCC^{\alpha}_{s,b}(k,\beta)\\
&\Rightarrow \quad zf'(z)\in UCC^{\alpha}_{s+1,b}(k,\beta)\\
&\Leftrightarrow \quad \Im^{\alpha}_{s,b}(zf(z))'\in UCC(k,\beta)\\
&\Leftrightarrow \quad z(\Im^{\alpha}_{s+1,b}f(z))'\in UCC(k,\beta)\\
&\Leftrightarrow \quad \Im^{\alpha}_{s+1,b}f(z)\in UQC(k,\beta)\\
&\Leftrightarrow \quad f(z)\in UQC^{\alpha}_{s+1,b}(k,\beta),
\end{aligned}$$

the proof is complete .

Now, we examine the closure properties of generalized Libera integral operator $L_c(f)(z)$. [10] defined by:

$$L_c(f)(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (f\in\mathbb{A},c>-1).$$

Theorem 3.5. *Let* $c > -\frac{k+\beta}{k+1}$, *and* $f \in \mathbb{A}$. *If*

$$\Im_{s,b}^{\alpha} f(z) \in SP(k,\beta) \quad so \quad is \quad L_c(\Im_{s,b}^{\alpha} f(z)).$$

Proof. From definition of $L_c(f)(z)$ and the linear operator $(\Im_{s,b}^{\alpha} f)(z)$, we have

$$z(\Im_{s,b}^{\alpha} L_c(f))'(z) = (c+1)\Im_{s,b}^{\alpha} f(z) - c(z(\Im_{s,b}^{\alpha} L_c(f))(z). \tag{3.12}$$

Setting

$$\frac{z(\Im_{s,b}^{\alpha} L_c f(z))'}{\Im_{s,b}^{\alpha} L_c f)(z)} = p(z),$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$, is analytic in $\mathbb{U}$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$.

From (3.12) we can write

$$p(z) = (c+1)\frac{(\Im_{s,b}^{\alpha} f(z)}{\Im_{s,b}^{\alpha} L_c f)(z)} - c. \tag{3.13}$$

By logarithmically differentiating both sides of the equation (3.13), we get

$$\frac{z(\Im_{s,b}^{\alpha} f(z))'}{\Im_{s,b}^{\alpha} f)(z)} = \frac{z(\Im_{s,b}^{\alpha} L_c f(z))'}{\Im_{s,b}^{\alpha} L_c f(z)} + \frac{zp'(z)}{p(z)+c},$$

$$\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha} f)(z)} = p(z) + \frac{zp'(z)}{p(z)+c}.$$

Therefore, the theorem follows by Lemma 2.1, since $\Re\{q_{k,\beta}(z) + c\} > 0$.

Which complete the proof of Theorem.

Theorem 3.6. *Let* $c > -\frac{k+\beta}{k+1}$, *and* $f \in \mathbb{A}$. *If*

$$\Im_{s,b}^{\alpha} f(z) \in UC(k,\beta) \quad so \quad is \quad L_c(\Im_{s,b}^{\alpha} f).$$

Proof. Consider the following

$$\begin{aligned} f(z) \in UC^{\alpha}_{s,b}(k,\beta) &\Leftrightarrow (\Im^{\alpha}_{s,b} f)(z) \in UC(k,\beta) \\ &\Leftrightarrow z(\Im^{\alpha}_{s,b} f(z))' \in SP(k,\beta) \\ &\Leftrightarrow \Im^{\alpha}_{s,b}(zf'(z)) \in SP(k,\beta) \\ &\Leftrightarrow zf'(z) \in SP^{\alpha}_{s,b}(k,\beta) \\ &\Rightarrow L_c(zf'(z)) \in SP^{\alpha}_{s,b}(k,\beta) \\ &\Leftrightarrow z(L_c f(z))' \in SP^{\alpha}_{s,b}(k,\beta) \\ &\Leftrightarrow (L_c f)(z) \in UC^{\alpha}_{s,b}(k,\beta), \end{aligned}$$

the proof is complete .

Theorem 3.7. *Let $c > -\frac{k+\beta}{k+1}$, and $f \in \mathbb{A}$. If*

$$\Im^{\alpha}_{s,b} f(z) \in UCC(k,\beta) \quad \textit{so} \quad \textit{is} \quad L_c(\Im^{\alpha}_{s,b} f).$$

Proof. Let $f(z) \in UCC^{\alpha}_{s,b}(k,\beta)$. Then, in view of the definition ,we can write

$$Re\left\{\frac{z(\Im^{\alpha}_{s,b} f(z))'}{\psi(z)}\right\} \prec q_{k,\beta}, \quad (z \in \mathbb{U}),$$

for some $\psi(z) \in SP(k,\beta)$. For g such that $\Im^{\alpha}_{s,b} g(z) = \psi(z)$,

$$Re\Big(\frac{z(\Im^{\alpha}_{s,b} f(z))'}{\Im^{\alpha}_{s,b} g(z)}\Big) \prec q_{k,\beta}. \tag{3.14}$$

New we set

$$\frac{z(\Im^{\alpha}_{s,b} L_c f(z))'}{\Im^{\alpha}_{s+1,b} L_c g(z)} = p(z), \tag{3.15}$$

Now from (3.12) we have

$$\begin{aligned} \frac{z(\Im^{\alpha}_{s,b} f(z))'}{\Im^{\alpha}_{s,b} g(z)} &= \frac{z(\Im^{\alpha}_{s,b} L_c(zf')'(z) + c\Im^{\alpha}_{s,b} L_c(zf'(z)}{z\Im^{\alpha}_{s,b} L_c g(z') + c\Im^{\alpha}_{s,b} L_c g(z)}, \\ &= \frac{\frac{z(\Im^{\alpha}_{s,b} L_c(zf')'(z))}{\Im^{\alpha}_{s,b} L_c g(z)} + \frac{c(\Im^{\alpha}_{s,b} L_c(zf'(z)))}{(\Im^{\alpha}_{s,b} L_c g)(z)}}{\frac{z(\Im^{\alpha}_{s,b} L_c(g))'(z)}{(\Im^{\alpha}_{s,b} L_c(g))(z)} - (1-b)}. \end{aligned} \tag{3.16}$$

Since $g(z) \in SP^{\alpha}_{s,b}(k,\beta)$,and by theorem 3.5, we have $(L_c g)(z) \in SP^{\alpha}_{s,b}(k,\beta)$. Letting

$$\frac{z((\Im^{\alpha}_{s,b} L_c g)(z))'}{(\Im^{\alpha}_{s,b} L_c g)(z)} = H(z).$$

Also, we can defined h by

$$z((\Im^{\alpha}_{s,b} L_c f)(z))' = (\Im^{\alpha}_{s,b} L_c g)(z)[h(z)], \tag{3.17}$$

differentiating both sides of (3.17) with respect to z, we get

$$\frac{z[z(\Im^{\alpha}_{s,b} L_c f(z))']'}{\Im^{\alpha}_{s,b} L_c g(z)} = zh'(z) + h(z)(H(z)). \tag{3.18}$$

Using (3.16) and (3.18), we obtain

$$\begin{aligned} \frac{z(\Im^{\alpha}_{s,b} f(z))'}{\Im^{\alpha}_{s,b} g(z)} &= \frac{h(z)H(z) + zh'(z) + ch(z)}{H(z)+c} \\ &= h(z) + \frac{zh'(z)}{H(z)+c}. \end{aligned} \tag{3.19}$$

This in conjunction with (3.14) leads to

$$h(z) + \frac{1}{H(z)+c} zh'(z) \prec q_{k,\beta}.$$

For letting $A = 0$ and $B(z) = \frac{1}{H(z)+c}$ we note that

$\Re\{B(z)\} > 0 \quad if \quad c > -\frac{k+\beta}{k+1}$. The above inequality satisfies the conditions required by Lemma 2.2 . so the proof is complete.

A similar argument yields

Theorem 3.8. *Let $c > -\frac{k+\beta}{k+1}$, and $f \in \mathbb{A}$. If*

$$\Im^{\alpha}_{s,b} f(z) \in UQC(k,\beta) \quad so \quad is \quad L_c(\Im^{\alpha}_{s,b} f).$$

Acknowledgement. The work is partially supported by UKM-ST-06-FRGS0244-2010,MOHE Malaysia.

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