University of Benghazi
Faculty of Education Almarj

جامعة بنغازي
كلية التربية – المرج
ISSN 2518-5845

Global Libyan Journal

المجلة الليبية العالمية

العدد السابع والأربعون / أبريل / 2020

On The Existence of A Unique Solution for Systems of Nonlinear Ordinary Differential Equations of Order m

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الملخص:

في هذا البحث نقدم نظرية تضمن وجود حلا وحيدا موضعيا لنظام من المعادلات التفاضلية العادية غير الخطية ذات رتبة m ، حيث (m) عددا صحيحا موجبا، ومعه شروطا ابتدائية.

بإثبات أن المؤثر غير الخطي المناظر لذلك النظام يكون مؤثرا تقليصيا (contractive) في فضاء متري جزئي E من فضاء بناخ B المكوّن من الدوال المتصلة والمحدودة والقابلة للاشتقاق على مجموعة الأعداد الحقيقية والمزود بنظيم موزون والذي يعرف باسم نظيم بيالسكي (Bielesck's norm) ، تم إثبات النظرية.

الكلمات المفتاحية: فضاء بناخ من الدوال المحدودة، نظيم بيالسكي، وجود حل وحيد بصورة عمومية ، نظام من المعادلات التفاضلية العادية غير الخطية من الرتبة m.

Abstract

In this paper, a theorem for local existence of a unique solution for a system of nonlinear ordinary differential equations (NODE) of order m has been stated and proved. The proof was done using contractive mapping theorem in a metric space subset of the Banach space consisting of the bounded differentiable functions on the real numbers and equipped with a Bielescki's type norm. As an application of the theory, an example has been given.

Index Terms : Banach space of bounded functions, Bielescki's type norm, Existence of *a* unique solution globally, System of nonlinear ordinary differential equations of order m.

I. INTRODUCTION

A theorem for the existence of a unique solution for nonlinear ordinary differential equations of order m was proved by A. A. Bojeldain [1], and a theorem for the existence of a unique solution for a system of nonlinear ordinary differential equations of the first order was proved by A. A. Bojeldain and S. E. Muhammed [2].

In this paper we study the system of nonlinear ordinary differential equations of order m having the general form:

$$X^{(m)}(t) = F(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) \qquad (1)$$

with the initial conditions,

$$X^{(j)}(a) = C^{j},\ j = 0, 1, 2, \dots, m-1, \quad X^{(0)}(a) = X(a) = C^{(0)}; \qquad (2)$$

where m is a finite positive integer, $t \geq a$ is a finite real number, and

$$X^{(m)}(t) = \begin{bmatrix} x_1^{(m)}(t) \\ x_2^{(m)}(t) \\ x_3^{(m)}(t) \\ \vdots \\ x_n^{(m)}(t) \end{bmatrix}, \qquad (3)$$

$$C^j = \begin{bmatrix} c_1^j \\ c_2^j \\ c_3^j \\ \vdots \\ c_n^j \end{bmatrix} \qquad for\ j = 0, 1, 2, \dots, m-1,\ n \text{ is a finite positive integer,} \qquad (4)$$

$$F\left(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)\right) = \begin{bmatrix} f_1(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) \\ f_2(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) \\ f_3(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) \\ \vdots \\ f_n(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) \end{bmatrix}, \qquad (5)$$

$$X^{(j)}(t) = (x_1^{(j)}(t), x_2^{(j)}(t), x_3^{(j)}(t), \dots, x_n^{(j)}(t)) \quad for\ j = 0, 1, 2, \dots, m-1. \qquad (6)$$

$$f_i\left(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)\right) = f_i\left(t, x_1(t), x_2(t), x_3(t), \dots, x_n(t);\ x'_1(t), x'_2(t), x'_3(t), \dots, x'_n(t);\ x''_1(t), x''_2(t), x''_3(t), \dots, x''_n(t); \dots, x_1^{(m-1)}(t), x_2^{(m-1)}(t), x_3^{(m-1)}(t), \dots, x_n^{(m-1)}(t)\right) \quad for\ i = 1, 2, 3, \dots, n. \qquad (7)$$

In other form the system is

$$x_i^{(m)}(t) = f_i(t, x_1(t), x_2(t), x_3(t), \dots, x_n(t);\ x'_1(t), x'_2(t), x'_3(t), \dots, x'_n(t);\ x''_1(t), x''_2(t), x''_3(t), \dots, x''_n(t); \dots, x_1^{(m-1)}(t), x_2^{(m-1)}(t), x_3^{(m-1)}(t), \dots, x_n^{(m-1)}(t))$$

$for\ i = 1, 2, 3, \dots, n.$

Since the system of (NODE) (1) with the initial condition (2) is equivalent to the integral equation:

$$X(t) = C^0 + \sum_{j=1}^{m-1} \frac{C^0 (t-a)^j}{j!} + \int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-1}} \int_a^{s_{m-2}} f_i\,(s, X(s), X'(s), X''(s), \dots, X^{(m-1)}(s))\, ds_m ds_{m-1} ds_{m-2} \dots ds_2 ds_1 \qquad (8)$$

we denote the right hand side of (8) by the nonlinear operator $Q(X)t$; then prove that this operator is contractive in a metric space E subset of the Banach space B of the class of continuous bounded functions $X(t) \in C^m(\mathbb{R})$ defined by:

$$B = \{(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)) : |t-a| < \infty, \left|X^{(j)}(t) - C^j\right| \le T^j < \infty,\ for\ j = 0, 1, 2, \dots, m-1\}, \qquad (9)$$

where $T^j = \max_{1 \le i \le n}\{T_i^j\}$, T_i^j are the upper bound for $\left|x_i^{(j)}(t) - c_i^{(j)}(t)\right|$ $for\ i = 1, 2, 3, \dots, n,\ j = 0, 1, 2, 3, \dots, m-1$, and B is equipped with the weighted norm:

$$\|X\| = \max_{|t-a|<\infty}\left(exp(-\nu L|t-a|)\sum_{j=0}^{m-1}\sum_{i=1}^{n}|x_i^{(j)}(t)|\right) \quad (10)$$

which is known as Bielescki's type norm. $\nu \geq 2$, $L = max(l, 1)$ are finite real numbers, where $l = max(l_i)$, l_i is the Lipschitz coefficient of $f_i(t, X(t), X'(t), X''(t), \ldots, X^{(m-1)}(t))$ $for\ i = 1, 2, 3, \ldots, n$ in $B1$ (a subset of the Banach space B given by (9)) defined by:

$$B1 = \{(t,\ X(t), X'(t), X''(t), \ldots, X^{(m-1)}(t))|\ |t-a| \leq T_m,\ |X^{(j)}(t) - C^j| \leq T^*\} \quad (11)$$

where $T^* = \min_{0\leq j\leq m-1}\{T^j\}$, and T_m a finite real number.

Concisely we will write $(t, \{X^{(j)}(t)\}_{j=0}^{m-1})$ instead of $(t,\ X(t), X'(t), X''(t), \ldots, X^{(m-1)}(t)$.

When the function F in the right hand side of (1) depends linearly on its arguments except t, then equation (1) is an m^{th} order system of linear ordinary differential equations and to prove the existence of a unique solution for it in $[a - T_m, a + T_m]$ one usually prove that component wise in a neighborhood $N_\delta(a)$ for $t \in [a, a+\delta]$, then doing the same steps of the proof for $t \in [a-\delta, a]$; after that use another theorem to show whether the solution do exist for all $t \in [a - T_m, a + T_m]$ or not as in [3]. By the theorem which we are going to state and prove in this paper one can easily prove the existence of a unique solution for m^{th} order nonlinear systems of ordinary differential equations on the general form (1) for all $t \in [a-\delta, a+\delta]$ directly in a very simple metric space E consisting of the functions $X(t) \in C^m[a - T_m, a + T_m]$, subset of the Banach space (9) [4], and equipped with the simple efficient norm (10) [5], which is a simple modification on the Bielescki's type norm $sup(e^{-r(t)}\ x(t))$ using in [6]. Moreover if the Lipschitz condition (12) is guaranteed to be satisfied in the Banach space (9), then the theorem guarantees the existence of a unique solution for $|t-a| < \infty$ in most cases and not in general as mentioned in [7].

Note that this theorem is valid for m^{th} order linear systems of ordinary differential equations as well.

II. MAIN THEOREM

Now, we are in a position to state and prove our main theorem.

Theorem: Let us have the system of (NODE) (1) with the initial conditions (2), and suppose that the function F in the right hand side of (1) is continuous and satisfies the Lipschitz condition:

$$\left|F\left(t,\ X(t), X'(t), X''(t), \ldots, X^{(m-1)}(t)\right) - F\left(t,\ Y(t), Y'(t), Y''(t), \ldots, Y^{(m-1)}(t)\right)\right| =$$
$$\left|F(t, \{X^{(j)}(t)\}_{j=0}^{m-1}) - F(t, \{Y^{(j)}(t)\}_{j=0}^{m-1})\right| \leq L\sum_{j=0}^{m-1}|X^{(j)}(t) - Y^{(j)}(t)| \quad (12)$$

in $B1$ given by (11); then the initial value problem (1) and (2) has a unique solution in the $(nm+1)$ dimensional metric space E (of the functions $X(t) \in C^m[a-\delta, a+\delta]) \subseteq B$ defined by:

$$E = \{(t,\ X(t), X'(t), X''(t), \ldots, X^{(m-1)}(t)): |t-a| \leq \delta,\ |X^{(j)}(t) - C^j| \leq T^*\} \quad (13)$$

such that $\delta = min(T_m, \frac{T^*}{M})$; where $M = \kappa\sum_{j=1}^{m}\frac{|t-a|^{j-1}}{j!}$, $\kappa = max(|C^j|,\ M^*)$,

$C^j = \max_{1\leq i\leq n}\{c_i^j\}$, $M^* = \max_{1\leq i\leq n}\{M_i\}$, $and\ M_i$ is the upper bound of $|f_i|$ in $B1$, i.e.:

$$\left|f_i\left(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)\right)\right| \le M_i \ \forall \left(t, X(t), X'(t), X''(t), \dots, X^{(m-1)}(t)\right) \in B1 \qquad (14)$$

Proof: Integrating both sides of (1) from a to t m-times and using the initial conditions (2), we obtain the system of integral equations (8).

To form a fixed point problem $X(t) = Q(X)t$ denote the right hand side of (8) by $Q(X)t$, and to apply the contraction mapping theorem we first show that $Q: E \to E$; then prove that Q is contractive in E.

We see that:

$$|Q(X)t - C^j| \le \sum_{j=1}^{m-1}\left|C^j \frac{(t-a)^j}{j!}\right| + \left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} |f_i(s, X(s), X'(s), X''(s), \dots, X^{(M-1)})|\, ds\, ds_{m-1}\, ds_{m-2} \dots ds_2 ds_1\right| \le \sum_{j=1}^{m-1}|C^j| \frac{|t-a|^j}{j!} + M^* \frac{|t-a|^m}{m!} \le \kappa \sum_{j=1}^{m} \frac{|t-a|^j}{j!}. \qquad (15)$$

Therefore:

$$|Q(X)t - C^0| \le \kappa \sum_{j=1}^{m} \frac{|t-a|^j}{j!} = \kappa |t-a| \sum_{j=1}^{m} \frac{|t-a|^{j-1}}{j!} \le M\delta \le T^* \qquad (16)$$

which means that $Q: E \to E$.

To prove that Q is contractive, we consider the difference:

$$|Q(X)t - Q(Y)t| =\le \left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} \left|F(s, \{X^{(j)}(s)\}_{j=0}^{m-1}) - -F(s, \{Y^{(j)}(s)\}_{j=0}^{m-1})\right| ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right|, \qquad (17)$$

or equivalently:

$$|q_i(X) - q_i(Y)| \le$$
$$\le \left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} |f_i(X) - f_i(Y)|\, ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right| \qquad (18)$$
$$for\ i = 1, 2, 3, \dots, n$$

which according to Lipschitz condition (12) yields:

$$|q_i(X) - q_i(Y)| \le$$
$$\le \left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} l_i \sum_{i=1}^{n} \sum_{j=0}^{m-1} \left|x_i^{(j)}(s) - y_i^{(j)}(s)\right| ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right| \qquad (19)$$

such that $Q(X) = [q_1(X(t)), q_2(X(t)), q_3(X(t)) \cdots q_n(X(t))]^T$.

Multiply the right hand side of (12) by $e^{-\nu L|t-a|}$ $e^{\nu L|t-a|}$ and get:

$$|Q(X)t - Q(Y)t| \le$$
$$\le \left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} L \sum_{i=1}^{n} \sum_{j=0}^{m-1} \left|x_i^{(j)}(s) - y_i^{(j)}(s)\right| e^{-\nu L|t-a|} \times \times e^{\nu L|t-a|} ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right| \qquad (20)$$

Inequality (20) leads to:

$$|Q(X)t - Q(Y)t| \le$$
$$\le L\left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} \left(\max_{|s-a|\le\delta} \left(e^{-\nu L|t-a|} \sum_{i=1}^{n} \sum_{j=0}^{m-1} \left|x_i^{(j)}(s) - y_i^{(j)}(s)\right|\right) \times \times e^{\nu L|t-a|}\right) ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right| =$$
$$= L\left|\int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} \left(\max_{|s-a|\le\delta} \left(e^{-\nu L|t-a|} \sum_{j=0}^{m-1} |X^{(j)}(s) - Y^{(j)}(s)|\right) \times \times e^{\nu L|t-a|}\right) ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1\right| \qquad (21)$$

According to (10), the norm definition, inequality (21) becomes:

$|Q(X)t - Q(Y)t| \leq$

$$\leq L\|X-Y\| \left| \int_a^t \int_a^{s_1} \int_a^{s_2} \dots \int_a^{s_{m-2}} \int_a^{s_{m-1}} e^{\nu L|t-a|} ds\, dS_{m-1}\, ds_{m-2} \dots ds_2\, ds_1 \right| \quad (22)$$

Manipulating the integrals in (22) we obtain the following inequality:

$$|Q(X)t - Q(Y)t| \leq L\|X-Y\| \left| \frac{1}{(\nu L)} \left(e^{\nu L|t-a|} - 1\right) - \sum_{j=1}^{m-1} \frac{|t-a|^j}{j!\,(\nu L)^{m-j}} \right| \leq$$

$$\leq L\|X-Y\| \left(\frac{1}{(\nu L)^m} \left(e^{\nu L|t-a|} - 1\right) + \sum_{j=1}^{m-1} \frac{(\nu L|t-a|)^j}{j!\,(\nu L)^m} \right) \leq$$

$$\leq L\|X-Y\| \left(\frac{1}{(\nu L)^m} \left(e^{\nu L|t-a|} - 1\right) + \sum_{j=1}^{\infty} \frac{(\nu L|t-a|)^j}{j!\,(\nu L)^m} \right) =$$

$$= \|X-Y\| \left(\frac{2}{(\nu L)^m} \left(e^{\nu L|t-a|} - 1\right) \right) \quad (23)$$

i.e.

$$|Q(X)t - Q(Y)t| \leq \|X-Y\| \left(\frac{2}{(\nu L)^m} \left(e^{\nu L|t-a|} - 1\right) \right) \quad (24)$$

Multiplying both sides of (24) by $e^{-\nu L|t-a|}$ leads to:

$$e^{-\nu L|t-a|}|Q(X)t - Q(Y)t| \leq \frac{2}{\nu(\nu L)^{m-1}} \left(1 - e^{-\nu L|t-a|}\right)\|X-Y\| \leq$$

$$\leq \frac{2}{\nu(\nu L)^{m-1}} \left(1 - e^{-\nu L\delta}\right)\|X-Y\| \quad (25)$$

The most right hand side of (25) is independent of t, thus it is an upper bound for its left hand side for any $|t-a| \leq \delta$; whence:

$$\max_{|t-a|\leq\delta}\left(e^{-\nu L|t-a|}|Q(X)t - Q(Y)t|\right) \leq \frac{2}{\nu(\nu L)^{m-1}}\left(1 - e^{-\nu L\delta}\right)\|X-Y\| \quad (26)$$

which, according to the norm definition (10), gives:

$$\|Q(X) - Q(Y)\| \leq \frac{2}{\nu(\nu L)^{m-1}}\left(1 - e^{-\nu L\delta}\right)\|X-Y\| \leq \left(1 - e^{-\nu L\delta}\right)\|X-Y\| \quad (27)$$

Noting that for finite $L \geq 1$, $\nu \geq 2$, and $m \geq 1$ we have

$$\frac{2}{\nu(\nu L)^{m-1}} \leq \frac{2}{\nu} \leq 1.$$

Since $0 < \left(1 - e^{-\nu L\delta}\right) < 1$; then $Q(X)t$ is contracting in E and thus has a unique solution for $t \in [a-\delta, a+\delta]$.

III. An Application

As an application of the result established in the previous section, in what follows we select the exact solutions:

$$\left.\begin{array}{r} x_1^* = t^3 \\ x_2^*(t) = e^t \\ x_3^*(t) = \sin t \end{array}\right\} \quad (28)$$

and construct the following system of nonlinear ordinary differential equations:

$$\frac{d^3 x_1(t)}{dt^3} = 6 - x_1^2 + 2x_1 t^6$$

$$\frac{d^3x_2(t)}{dt^3} = 2e^t - x_2 \qquad (29)$$
$$\frac{d^3x_3(t)}{dt^3} = 2x_3 \sin t - x_3^2 + \sin^2 t$$

If we set $t = 0$ in (28), we get

$$x_1^*(0) = 0, \quad \frac{d}{dt}x_1^*(0) = 0, \quad \frac{d^2}{dt}x_1^*(0) = 0$$
$$x_2^*(0) = 1, \quad \frac{d}{dt}x_2^*(0) = 1, \quad \frac{d^2}{dt}x_2^*(0) = 1 \qquad (30)$$
$$x_3^*(0) = 0, \quad \frac{d}{dt}x_3^*(0) = 1, \quad \frac{d^2}{dt}x_3^*(0) = 0$$

As the initial conditions to (29).

Selecting positive finite real numbers T_i^j $for\ i = 1,2,3,\ j = 0,1,2$, we find that $\left|x_i^{(j)} - c_i^j\right| \le T_i^j$ leads to

$|x_1| \le T_1^0,\ |x_1'| \le T_1^1,\ |x_1''| \le T_1^2,\ |x_2| \le T_2^0 + 1,\ |x_2'| \le T_2^1 + 1,\ |x_2''| \le T_2^2 + 1,\ |x_3| \le T_3^0,\ |x_3'| \le T_3^1 + 1,$ $and\ |x_3''| \le T_3^2.$

In $B1$ we have

$$|f_1(t,X,X',X'')| = |6 - x_1^2 + 2x_1t^6| \le 6 + (T_1^0)^2 + 2T_1^0(T_m)^3 + (T_m)^6,$$
$$|f_2(t,X,X',X'')| = |2e^t - x_2| \le 2e^{T_m} + T_2^0 + 1,$$

and

$$|f_3(t,X,X',X'')| = |2x_3 \sin t - x_3^2 + \sin^2 t| \le 2T_3^0 + (T_3^0)^2 + 1,$$

i.e.

$$M_1 = 6 + (T_1^0)^2 + 2T_1^0(T_m)^3 + (T_m)^6, \quad M_2 = 2e^{T_m} + T_2^0 + 1, \text{and} \quad M_3 = e^{T_m} + T_2^0 + 1.$$

Hence $M^* = max\{6 + (T_1^0)^2 + 2T_1^0(T_m)^3 + (T_m)^6,\ 2e^{T_m} + T_2^0 + 1,\ e^{T_m} + T_2^0 + 1\}$ (31)

Next, we check the Lipschitz condition for $f_1, f_2,\ and\ f_3$:

$$|f_1(t,X,X',X'') - f_1(t,Y,Y',Y'')| = |(6 - x_1^2 + 2x_1t^6) - (6 - y_1^2 + 2y_1t^6)| \le$$
$$\le (2T_1^0 + 2(T_m)^3)(|X - Y| + |X' - Y'| + |X'' - Y''|),$$

therefore f_1 satisfies Lipschitz condition (12) in $B1$ with Lipschitz coefficient

$$l_1 = 2T_1^0 + 2(T_m)^3,$$
$$|f_2(t,X,X',X'') - f_2(t,Y,Y',Y'')| = |(2e^t - x_2) - (2e^t - y_2)| \le$$
$$\le (|X - Y| + |X' - Y'| + |X'' - Y''|), \qquad (32)$$

i.e., f_2 satisfies Lipschitz condition (12) in $B1$ with Lipschitz coefficient $l_2 = 1$,

$$|f_3(t,X,X',X'') - f_3(t,Y,Y',Y'')| = |(2x_3 \sin t - x_3^2 + \sin^2 t) - (2y_3 \sin t - y_3^2 +$$
$$+\sin^2 t)| \le 2(1 + T_3^0)(|X - Y| + |X' - Y'| + |X'' - Y''|) \qquad (33)$$

whence f_3 satisfies Lipschitz condition (12) in $B1$ with Lipschitz coefficient

$$l_3 = 2(1 + T_3^0). \qquad (34)$$

Therefore,

$$L = max(2T_1^0 + 2(T_m)^3, 1, 2(1 + T_3^0)). \qquad (35)$$

Now $|C^j| = 1,\ \kappa = M^*,\ and\ M \le M^*\left(1 + \frac{1}{2}T_m + \frac{1}{6}(T_M)^2\right).$

These calculations lead to:

$$\delta = min\left(T_m, \frac{T^*}{M^*\left(1+\frac{1}{2}T_m+\frac{1}{6}(T_m)^2\right)}\right)$$

where $T^* = \min_{0\le j\le 2}\{T^0, T^1, T^2\}$.

So, we find that the unique solution exists in the interval $|t-a| \le \delta$

Conclusion

We see that the contraction coefficient $0 < \left(1-e^{-\nu L\delta}\right) < 1$ for any finite $\delta > 0$ which means that the solution for the problem under consideration is, in fact, guaranteed globally for
$|t-a| \le T_m$ and not only locally for $|t-a| \le \delta$. Moreover; in most cases; if the function F in the right hand side of (1) is continuous and satisfies Lipschitz condition in Banach space (9) with finite positive Lipschitz coefficient then the theorem is proved for t in any interval I of finite length because the contraction coefficient will be positive and less than
$\left(1- -e^{-\nu L\mu(I)}\right) < 1$; where $\mu(I)$ is the measure of the interval I.

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